

Competitive Equilibrium in Position Markets with Externalities

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October 28, 2026

ABSTRACT: We study competitive equilibrium in one-sided assignment problems in which an agent's payoff depends not only on his own position but also on those of all other agents, and no two agents can occupy the same position. A definition of competitive equilibrium requires an answer to the question: When an agent moves to an occupied position, where does the occupant go? To address this, we introduce two concepts. In a swap-CE, an agent moves by forcing an exchange of positions; in a displacement-CE, an agent's move preserves the relative order of all others. We compare these concepts with pairwise stability and apply them to three settings: satisficing, sequential elimination, and serial dictatorship.

KEYWORDS: assignment problems; competitive equilibrium; externalities; pairwise stability; swap competitive equilibrium; displacement competitive equilibrium; satisficing; sequential elimination; and serial dictatorship.

JEL CLASSIFICATION: C78, D50, D62

We thank Kemal Yildiz for a helpful discussion and comments. Michael Richter is a full-time faculty member at Baruch College and part-time at Royal Holloway. He gratefully acknowledges funding from the Leverhulme Trust.

1. Introduction

In a competitive equilibrium setting, given prices, each agent chooses an alternative and ignores the possibility that his choice may conflict with the choice of others. Without externalities, this is a reasonable presumption, as he does not care about their choices. However, once an agent cares about the choices of others and his choice may conflict with theirs, a definition of competitive equilibrium necessitates a specification of the consequences of his choice on other agents' choices.

In this paper, we confront this dilemma while modeling the market for positions in a sequence where an agent's value of a position depends on the positions of the other agents. For example, when an agent is standing in line waiting to pick an object from a collection, the identities of the agents preceding him determine what is left for him to choose from. In such situations, the definition of competitive equilibrium requires a specification of how an agent's move affects the positions of other agents.

Let us be more concrete. Consider a setting with a set A of agents and an equally sized set of positions Π . Let $x=(x_i)_{i \in \Pi}$ be an assignment of agents to positions, where x_i is the unique agent occupying position i , and let $p=(p_i)_{i \in \Pi}$ be a vector of prices, one for each position. Given the assignment x and prices p , agent x_i obtains the payoff $U_{x_i}(x)-p_i$. If he considers moving to position j , he expects the resulting assignment to be some y such that $y_j=x_i$, and in equilibrium he expects to obtain the payoff $U_{x_i}(y)-p_j$, which does not exceed $U_{x_i}(x)-p_i$. But what is this assignment y ?

The rule determining y is an essential element that must be specified in order to extend the definition of competitive equilibrium (CE). We consider two rules: in a *swap-CE*, y results from a *swap* of x_i and x_j ; and in a *displacement-CE*, when an agent x_i moves to position j , the remaining agents retain their relative positions in the assignment.

Notice the difference between our approach and the standard general-equilibrium definition of competitive equilibrium with externalities which is sometimes referred to as a Nash-Walras equilibrium (see Arrow and Hahn (1971), Ghosal and Polemarchakis (1997), and del Mercato and Platino (2017)). There, an equilibrium is a feasible vector of choices and a price vector such that every individual optimizes his choice holding fixed both the prices and the choices of the other agents. When evaluating a different action, an agent does not take into account that he disturbs the feasibility of their choices, nor what will be the resulting feasible allocation.

Our environment relates to Shapley and Shubik’s (1971) classical one-sided assignment model where a “house” in their model corresponds to a “position” in ours. In their competitive equilibrium notion, agents do not care who occupies the other positions and do not consider what happens to other agents if they take a different position than their equilibrium choice. Most of the literature on assignment problems with externalities allows the matches of the other agents to affect an agent’s preferences but ignores the downstream effects of an agent taking a different match (see, for example, Pycia and Yenmez (2023)).¹

The closest to our approach is a strand of the literature initiated by Sasaki and Toda (1996), who consider pairwise stable assignments in a two-sided matching problem where the value of a matched pair also depends on who else is matched with whom. They propose a notion of conjectural equilibrium, in which, when a pair considers deviating, it has a systematic conjecture about how the rest of the matching will adjust (see also Gudmundsson and Habis (2017) and Imamura, Konishi, and Pan (2023)).

We apply our CE concepts and the notion of pairwise stability to three settings.

1. Satisficing:

The positions are locations in a queue. A decision maker goes through the agents according to their order in the queue until he finds an agent who is satisfactory. If no satisfactory agent is found, then he does not choose anyone. Agents differ in the probability that they are found satisfactory, and all wish to maximize their chance of being selected. The value of an agent’s position depends on the qualities of all the agents who are ahead of him in the queue. Manzini and Mariotti (2014) consider an attention model in which, for a given order, an alternative’s probability of being picked can be calculated in an identical fashion.

2. Sequential Elimination:

The agents differ in their “power”. When a stronger agent meets a weaker one, the stronger wins most of the time, but the weaker agent can still win with some positive probability. The agents compete in a gauntlet tournament: the agents are

¹Note that although Gale (1984) studies an assignment model with externalities, his analysis considers only the dependence of preferences on prices.

assigned to positions and are compared sequentially in pairs, from right to left. The loser of a match leaves the list, and the winner is matched with the next agent in line. The selected agent is the one who wins the final match. The probability that an agent is selected in a given position depends not only on the identities of the agents to his left (whom he will have to confront) but also on the probability that his first opponent (coming from the right) is stronger or weaker than him. This contrasts with the preceding example, in which only the agents ahead of him mattered.

3. Serial Dictatorship:

Agents are called sequentially to choose an object from a finite collection (a draft). An agent's utility is determined by the object he receives; hence, an agent's value of a position depends on his preferences over objects and on the identities of the agents in earlier positions.

In all three settings, a position has no intrinsic value. An agent's payoff is not determined by his position alone; it also depends on how the other agents are positioned.

The first two settings are related to the literature on random choice because they can be viewed as random procedures for selecting an alternative from a fixed choice set. However, our objective is to study equilibrium assignments and to clarify the implications of alternative definitions of competitive equilibrium.

2. General framework

There is a set $A = \{a_1, \dots, a_n\}$ of *agents* and an equal-sized set $\Pi = \{1, \dots, n\}$ of *positions*. An *assignment* is a one-to-one function between agents and positions. We denote a generic assignment by $x = (x_i)_{i \in \Pi}$, where x_i is the agent who occupies position i . Generic agents are denoted by a and b , and generic positions by i and j . Each part of the paper will specify a function $U_a(x)$ which is the transferable payoff of agent a under the assignment x . Given an assignment x , we consider two types of systematic changes to the assignment.

A *displacement assignment* denoted by $x^{i \rightarrow j}$ is obtained when an agent moves from position i to position j , while the other agents retain their relative order. We say that agent x_i *displaces* x_j . Formally,

if $j < i$, then $x^{i \leftarrow j} = (\dots, x_{j-1}, \overbrace{x_j, \dots, x_i}^{\text{swap}}, \dots)$

if $j > i$, then $x^{i \leftarrow j} = (\dots, \overbrace{x_i, \dots, x_j}^{\text{swap}}, x_{j+1}, \dots)$

A *swap assignment*, denoted $x^{i \leftarrow j}$, is obtained by *swapping* x_i and x_j . Formally,

$$x^{i \leftarrow j} = (\dots, x_{i-1}, \overbrace{x_i, x_{i+1}, \dots, x_{j-1}, x_j}^{\text{swap}}, x_{j+1}, \dots)$$

We consider three solution concepts. First, a standard one:

Definition. An assignment x is **pairwise stable** if there are no two agents x_i and x_j such that

$$U_{x_i}(x^{i \leftarrow j}) + U_{x_j}(x^{i \leftarrow j}) > U_{x_i}(x) + U_{x_j}(x).$$

We use the sum of the two utilities as the criterion for a profitable pairwise deviation.

The two central innovations of the paper are adaptations of CE. In both of them, if an agent x_i moves to position j , then he sells position i for p_i and buys position j for p_j . In a displacement-CE, the relative order of the other agents is preserved when x_i sneaks into position j . In a swap-CE, the agent in position j is forced to move to position i . Formally,

Definition. A **displacement-CE** is an assignment x and a price vector p for which there is no agent x_i and position j such that

$$U_{x_i}(x^{i \leftarrow j}) - p_j > U_{x_i}(x) - p_i.$$

Definition. A **swap-CE** is an assignment x and a price vector p for which there is no agent x_i and position j such that

$$U_{x_i}(x^{i \leftarrow j}) - p_j > U_{x_i}(x) - p_i.$$

Without loss of generality, we normalize prices so that the minimum price is always 0.

Any swap-CE assignment is pairwise stable. (In Section 5, we will see that this is not true for displacement-CE.) To see why, note that for every pair of positions i and j ,

$$\left. \begin{array}{l} U_{x_i}(x) - p_i \geq U_{x_i}(x^{i \leftarrow j}) - p_j \\ U_{x_j}(x) - p_j \geq U_{x_j}(x^{i \leftarrow j}) - p_i \end{array} \right\} \implies U_{x_i}(x) + U_{x_j}(x) \geq U_{x_i}(x^{i \leftarrow j}) + U_{x_j}(x^{i \leftarrow j}).$$

Discussion: The three solution concepts discussed in this paper reflect different types of competitive processes. Pairwise stability captures the possibility that two agents agree to swap positions in exchange for a monetary transfer negotiated between them. The consent of both parties is required for the swap. Swap-CE reflects a situation in which an agent can force an exchange by paying the difference between the price of the position he moves to and the price of his original position. Displacement-CE reflects the possibility of a unilateral move in which an agent can sneak into a line by paying the difference between the price of the new position and that of the old one.

3. Satisficing

The agents are interviewed sequentially from left to right and all want to be chosen. The first agent who is found satisfactory is chosen. The probability that an agent a is judged to be satisfactory is α_a , where $1 > \alpha_a > 0$. For simplicity, we will often write α_i instead of α_{a_i} and index the agents such that $\alpha_1 > \dots > \alpha_n$. If no agent is found satisfactory, then none is chosen. Given an assignment $x = (x_1, \dots, x_n)$, the payoff of agent x_i is

$$U_{x_i}(x) = \prod_{k=1}^{i-1} (1 - \alpha_{x_k}) \alpha_{x_i}.$$

The following claim states that the pairwise-stable assignments are $(a_1, \dots, a_n)$ and any assignment which can be obtained by non-overlapping swaps of adjacent agents. Thus, $x = (a_1, a_2, a_3, a_4, a_5, a_6)$ and $(a_1, a_3, a_2, a_4, a_6, a_5)$ are pairwise stable, whereas $(a_1, a_4, a_2, a_3, a_5, a_6)$ is not.

The intuition is as follows. In any assignment, swapping two consecutive agents does not affect their joint probability of winning. For non-consecutive agents, however, this probability is higher when the stronger agent is interviewed first, since the later agent is less likely to be reached: not only must all preceding agents fail, but so must those between the two agents.

Given any assignment x , denote the probability that x_i is interviewed by

$$B_i(x) = \prod_{k=1}^{i-1} (1 - \alpha_{x_k})$$

We suppress the dependence of B_i on x when the assignment is clear.

Proposition 1. *An assignment x is pairwise stable iff $\alpha_{x_i} > \alpha_{x_j}$ whenever $i+1 < j$. That is, the assignment must be decreasing in strength, except for possible adjacent inversions.*

Proof. Consider an assignment x and a pair of non-consecutive agents $a=x_i$ and $b=x_j$ where $i+1 \leq j$. The swap increases the pair's joint probability of winning iff

$$B_i \alpha_a + B_j \alpha_b < B_i \alpha_b + B_j \frac{(1-\alpha_b)}{(1-\alpha_a)} \alpha_a.$$

Equivalently, $(B_{i+1} - B_j)(\alpha_b - \alpha_a) > 0$. Since B_i is strictly decreasing in i , the inequality holds iff $i+1 < j$ and $\alpha_b > \alpha_a$. $\square$

We turn to characterizing the displacement-CE and swap-CE assignments. In the current setting, the two concepts are identical because an agent's probability of being selected depends only on the set of agents ahead of him, and not on their order. When an agent in position i takes position j , the two concepts differ in the order of agents before x_i but not in the set of agents before x_i .

Proposition 2. *A pair (x, p) constitutes a displacement/swap-CE iff $x=(a_1, \dots, a_n)$ and $p_i - p_{i+1} = B_i \alpha_{x_i} \alpha_{x_{i+1}}$ (when $n=2$, $x=(a_2, a_1)$ is also a CE assignment).*

Proof. In any equilibrium, prices must satisfy $p_1 > \dots > p_n$, since otherwise an agent in a later position would prefer to move to an earlier and weakly cheaper position.

Fix an assignment x and a price vector p . Neither x_i nor x_{i+1} wants to take the other's position iff:

$$B_i \alpha_{x_i} - p_i \geq B_i(1 - \alpha_{x_{i+1}}) \alpha_{x_i} - p_{i+1} \text{ and } B_i(1 - \alpha_{x_i}) \alpha_{x_{i+1}} - p_{i+1} \geq B_i \alpha_{x_{i+1}} - p_i.$$

These inequalities hold iff $B_i \alpha_{x_i} \alpha_{x_{i+1}} = p_i - p_{i+1}$ for every $i < n$.

Necessity: Let (x, p) be an equilibrium. Given the normalization $p_n = 0$, the equilibrium prices are uniquely determined by the equation $p_i - p_{i+1} = B_i \alpha_{x_i} \alpha_{x_{i+1}}$. For $i \leq n-2$, since x_i does not want to move to position $i+2$, it must be that

$$B_i \alpha_{x_i} \alpha_{x_{i+1}} + B_i(1 - \alpha_{x_i}) \alpha_{x_{i+1}} \alpha_{x_{i+2}} \leq B_i \alpha_{x_i} \alpha_{x_{i+1}} + B_i(1 - \alpha_{x_{i+1}}) \alpha_{x_i} \alpha_{x_{i+2}} \Leftrightarrow$$

$$(1 - \alpha_{x_i}) \alpha_{x_{i+1}} \leq (1 - \alpha_{x_{i+1}}) \alpha_{x_i} \Leftrightarrow \alpha_{x_{i+1}} \leq \alpha_{x_i}.$$

Assume $n \geq 3$. An analogous calculation shows that agent x_n does not have a profitable move from position n to $n-2$ iff $\alpha_{x_{n-1}} \geq \alpha_{x_n}$. Thus, for $n \geq 3$, it must be that $\alpha_{x_1} > \alpha_{x_2} > \dots > \alpha_{x_n}$.

Sufficiency: Let $x=(a_1,\dots,a_n)$ (or $n=2$ and $x=(a_2,a_1)$) and $p_i-p_{i+1}=B_i\alpha_i\alpha_{i+1}$ for every $i<n$. Recall that since $x_i=a_i$, it holds that $\alpha_{x_i}=\alpha_{a_i}=\alpha_i$. Every agent is indifferent between his own position and any adjacent position. To see that agent x_i does not benefit from moving from position i to position $k>i+1$, it suffices to verify that $B_l\alpha_l\alpha_{l+1}<B_{l+1}\frac{1}{1-\alpha_i}\alpha_i\alpha_{l+1}$ for each l from $i+1$ to $k-1$, which would mean that at every stage from l to $l+1$, agent i gains less than agent l would (who is indifferent between these positions). The inequality is equivalent to $(1-\alpha_i)\alpha_l<(1-\alpha_l)\alpha_i$ which is equivalent to $\alpha_l<\alpha_i$. An analogous argument applies for a move from i to $k<i-1$. $\square$

4. The sequential-elimination procedure

A decision maker faces an ordered sequence of agents and chooses one through *sequential elimination*. Starting from the right, he compares the last two agents, keeps the winner, and compares that winner with the next agent to the left. He continues until all agents have been considered.² Formally, given an assignment $x=(x_1,\dots,x_n)$, compare x_n with x_{n-1} and denote the winner by w_{n-1} . Then compare w_{n-1} with x_{n-2} , obtaining w_{n-2} , and continue in this way until the final winner w_1 is determined. Denote $w_n=x_n$.

We identify the assignments x and $x^{n\leftarrow n-1}$, since they induce exactly the same comparisons. Therefore, in this section, we do not allow swaps or displacements among positions $n-1$ and n .

The agents are strictly ordered by strength $a_1\succ a_2\succ\dots\succ a_n$. In any pairwise comparison between two agents, the stronger wins with probability $1-\varepsilon$ and the weaker wins with probability ε , where $\varepsilon\in(0,\frac{1}{2})$. Strength is purely ordinal and the upset probability is ε independently of the gap in the ranking between the two agents. Finally, the agents' utility is their probability of winning. Define $U_a(x)$ to be the probability that a is the final winner under assignment x .

²In the context of choice from lists, such a procedure was analyzed in a deterministic setting by Rubinstein and Salant (2006) and in a stochastic setting by Yildiz (2016). Both papers axiomatically characterize choice functions consistent with this procedure by varying the choice sets. In contrast, in the current paper, the choice set is fixed, and the alternatives are agents competing for positions in a list.

Notation: Given a fixed assignment x :

Let q_i^j denote the probability that $x_i \prec w_j$ for any $i < j \leq n$.

For any agent a , let $\alpha(q) := q + (1-q)(1-\varepsilon)$ be the probability that the winner of a contest is stronger than a when (i) one contestant is stronger than a , and (ii) the other contestant is stronger than a with probability q .

Similarly, for any agent a , let $\beta(q) := q(1-\varepsilon)$ be the probability that the winner of a contest is stronger than a when (i) one contestant is weaker than a , and (ii) the other contestant is stronger than a with probability q .

The importance of these functions is that the probability q_i^{i+1} that the first contestant that an agent x_i faces can be expressed as a composition of α and β by reading the sequence of agents to the right of him and using α when $x_i \prec x_j$ and using β when $x_i \succ x_j$. Clearly $\alpha(q) > \beta(q)$. Also, for every $q \in [0,1]$, $\alpha \circ \beta(q) > \beta \circ \alpha(q)$, that is, for any agent, holding an assignment fixed except for two consecutive agents to the right of him, he has a higher probability of facing an opponent stronger than him if those two consecutive agents are ordered with the stronger rather than the weaker on the left.

For $i < n$, agent x_i 's overall probability of winning can be written as:

$$U_{x_i}(x) = \underbrace{(q_i^{i+1}\varepsilon + (1-q_i^{i+1})(1-\varepsilon))}_{\text{Pr } x_i \text{ is chosen over } w_{i+1}} \underbrace{\varepsilon^{|\{j|j < i \text{ and } x_j \succ x_i\}|} (1-\varepsilon)^{|\{j|j < i \text{ and } x_j \prec x_i\}|}}_{\text{Pr } x_i \text{ is chosen over } x_{i-1}, \dots, x_1}$$

The utility of the last agent, x_n , is calculated using only the right term as there is no agent to the right of him. Given the above formula, and the explanation of how q_i^{i+1} is calculated, we immediately have the following lemma:

Lemma 1. *The probability that an agent x_i wins depends only on:*

- (i) *The sequence indicating for each position to the right of i whether the agent in that position is stronger or weaker than x_i and*
- (ii) *The number of agents stronger than x_i who appear to the left of x_i .*

4.1 Properties

We first prove that in any assignment, any agent benefits by moving to the left, either by swapping or displacing.

Lemma 2 (Moving to the left is beneficial). *For any assignment x , and positions $j < i$, it holds that $U_{x_i}(x^{i \leftarrow j}) > U_{x_i}(x)$ and $U_{x_i}(x^{i \rightarrow j}) > U_{x_i}(x)$.*

Proof. In the assignment x , agent x_i wins if and only if he defeats all agents $x_j, \dots, x_{i-1}, w_{i+1}$ and then all agents $x_1, \dots, x_{j-1}$. In the assignment $x^{i \leftarrow j}$, agent x_i only has to defeat the survivor among $x_{j+1}, \dots, x_{i-1}, x_j, w_{i+1}$, and then all agents $x_1, \dots, x_{j-1}$. Thus, agent x_i moving to position j strictly increases his probability of winning. The argument for displacements is analogous. $\square$

Lemma 3 states that an agent k benefits when an agent j who is stronger swaps with an agent i who is to the right of both j and k .

Lemma 3. *Let $k < n-1$. Let $i, j \neq k$ be such that $j < i, n-1$. If $x_j \succ x_i$, then $U_{x_k}(x^{i \leftarrow j}) \geq U_{x_k}(x)$ with a strict inequality if and only if $x_j \succ x_k \succ x_i$ and $k < i$.*

Proof. If both x_i and x_j are to the left of k , then x_k 's probability is unaffected by their swap (Lemma 1).

If x_k is not $\succ$ -sandwiched, then by Lemma 1, the swap of x_i and x_j does not matter for x_k . Otherwise, $x_j \succ x_k \succ x_i$. Two cases remain:

Case (i) $k < j$. In this case, the swap can be decomposed as a sequence of swaps of consecutive agents, either moving x_j to the right or x_i to the left. All swaps are either between two agents stronger than x_k , two agents weaker than x_k , or between a stronger agent on the left and a weaker agent to the right. Only swaps of the last type affect q_k^{k+1} and they decrease it since $\alpha \circ \beta(q) > \beta \circ \alpha(q)$ for every $q \in [0, 1]$. Furthermore, at least one such swap exists (when x_j crosses x_i).

Case (ii) $j < k < i$. Among the agents that come to the left of x_k , let s be the number that are stronger than x_k and let w be the number of agents that are weaker. $U_{x_k}(x)$ is at most $[1-\varepsilon]\varepsilon^s(1-\varepsilon)^w$ (the upper bound on the probability that x_k beats w_{k+1} times the probability that he beats everyone to his left) while $U_{x_k}(x^{i \leftarrow j})$ is at least $[\varepsilon]\varepsilon^{s-1}(1-\varepsilon)^{w+1}$ which are equal. At least one of the inequalities is strict because it is impossible that everyone to the right of x_k was weaker than x_k before the swap and everyone to the right of x_k is stronger than x_k after the swap. $\square$

The next lemma shows that swapping a stronger agent on the right with a weaker agent on the left improves their joint probability if there is an agent between them in strength and between or to the left of them in position.

Lemma 4. *Consider an assignment x and two agents x_i, x_j such that $j < i$ and $x_j \prec x_i$. Then $U_{x_i}(x^{i \leftrightarrow j}) + U_{x_j}(x^{i \leftrightarrow j}) \geq U_{x_i}(x) + U_{x_j}(x)$ with strict inequality if and only if there is an x_k such that $x_j \prec x_k \prec x_i$ and $k < i$ and $k < n-1$.*

Proof. Consider a swap between x_i and x_j . The swap leaves unchanged the winning probabilities of all agents except the two swapping agents and any agent x_k who is between them in strength and to the left of i . By Lemma 3, x_k 's probability of winning strictly decreases. Therefore the joint probability of winning for the two swapping agents weakly rises and strictly so when the condition holds. $\square$

4.2 Characterization of equilibrium

Before characterizing the equilibrium notions, we show that for the sequential elimination model, not only are the swap-CE assignments pairwise stable, but also every displacement-CE assignment is pairwise stable.

Lemma 5. *Every displacement-CE assignment is pairwise stable.*

Proof. Let (x, p) be a displacement-CE. If x is not pairwise stable, then by Lemma 4, there are $i < j$, such that $x_i \prec x_j$ for whom $U_{x_i}(x^{i \leftrightarrow j}) + U_{x_j}(x^{i \leftrightarrow j}) > U_{x_i}(x) + U_{x_j}(x)$.

By Lemma 1, $U_{x_i}(x^{i \leftrightarrow j}) = U_{x_i}(x^{i \rightarrow j})$. Furthermore, $U_{x_j}(x^{i \leftrightarrow j}) \leq U_{x_j}(x^{j \rightarrow i})$ because for every $q \in [0, 1]$, $\alpha(\beta(q)) > \beta(\alpha(q))$, so it is better for x_j to have the weaker x_i nearer to him (when $j \rightarrow i$) than farther from him (when $i \leftrightarrow j$). Therefore, $U_{x_i}(x^{i \leftrightarrow j}) - p_j + U_{x_j}(x^{j \rightarrow i}) - p_i > U_{x_i}(x) - p_i + U_{x_j}(x) - p_j$ and at least one of i, j has a strictly profitable move. $\square$

The next three propositions characterize our three equilibrium notions: pairwise stability, displacement-CE, and swap-CE. They use the following notion of block decomposition.

An assignment is *block-decomposable* if it can be partitioned into a sequence of blocks of agents $[B_1, \dots, B_L]$ such that:

- (i) all agents in block B_k are stronger than all agents in block B_l whenever $k < l$; and
- (ii) each block is either a singleton or a “special block” of the form $(a_t, a_s, a_{s+1}, \dots, a_{t-1})$ where $s < t$.

For example, the assignment $(a_3, a_1, a_2, a_4, a_8, a_5, a_6, a_7)$ is block-decomposable with blocks $B_1=(a_3, a_1, a_2)$, $B_2=(a_4)$, and $B_3=(a_8, a_5, a_6, a_7)$.

Proposition 3. *An assignment is pairwise stable if and only if it is block-decomposable.*

Proof. For $n=3$ and given the identification of the last two positions, all assignments are block-decomposable: $[a_1][a_2][a_3]$, $[a_2, a_1][a_3]$, and $[a_3, a_1, a_2]$ and are pairwise stable, since swapping any two agents does not affect the other agent’s probability of winning.

For the rest of the proof, suppose $n \geq 4$:

Necessity. Let x be a pairwise stable assignment. We proceed by induction on the number of agents. Assume first that $x_1 = a_1$. Let $y = (x_2, \dots, x_n)$. For any agent $a \neq a_1$, $U_a(x) = \varepsilon U_a(y)$. The assignment y is pairwise stable because any improving swap in y is also improving in x . Thus y has a block representation, $B_1, \dots, B_L$, and therefore x has the block representation $[a_1], B_1, \dots, B_L$.

Otherwise, $x_1 = a_k \neq a_1$. We now show that $(x_2, \dots, x_k) = (a_1, \dots, a_{k-1})$.

First, there is no $j < k-1$ such that a_{j+1} is to the left of a_j , since then by Lemma 4, a_j and a_k could profitably swap ($a_k \prec a_{j+1} \prec a_j$ and a_{j+1} is to the left of a_j).

Second, there is no agent $a_m \prec a_k$ to the left of a_{k-1} because then, by Lemma 4, a swap of a_{k-1} and a_m would be profitable. Therefore, we can define $B_1 = (a_k, a_1, \dots, a_{k-1})$ and the desired block decomposition is obtained by applying the induction hypothesis to the remaining agents.

Sufficiency. Let x be a block-decomposable assignment. By Lemma 4, a profitable swap must involve two agents in positions $i < j$ such that $x_i \prec x_j$. By block-decomposability, x_i is the first element of a block and x_j belongs to the same block. All agents $\succ$ -sandwiched between x_i and x_j are to the right of x_j . But then, by Lemma 4, the swap does not change the sum of their probabilities of winning. $\square$

By Lemma 5 and Proposition 3, for this model, it holds that all displacement-CE

assignments and all swap-CE assignments are block decomposable. The next lemma proves that for both concepts, all blocks are either singletons or doubletons.

Lemma 6. *Suppose $n \geq 4$. Every swap-CE assignment and every displacement-CE assignment is block-decomposable with all blocks of size 1 or 2, except the last block which can be of size 3.*

Proof. Let (x, p) be a swap-CE or a displacement-CE. If there is a non-final block of size 3 or greater, we can assume without loss of generality that it is the first block. Then, $(x_1, x_2, x_3) = (a_i, a_1, a_2)$ and let $q = \Pr(w_4 \succ a_i)$. The following inequalities are necessary:

(i) Since a_i does not want to swap with a_2 :

$$q\epsilon + (1-q)[\epsilon^2(1-\epsilon) + (1-\epsilon^2)\epsilon] - p_1 \geq q\epsilon^3 + (1-q)(1-\epsilon)\epsilon^2 - p_3$$

$$\text{and thus } \epsilon - \epsilon^3 \geq p_1 - p_3.$$

Notice that the move of a_i to the right gives the same utility under either concept because the agent cares about the set of agents to the left of him, but not their order (Lemma 1) and thus the equilibrium conditions for rightward moves are the same under the two concepts.

(ii) Since a_2 does not want to swap with a_1 , it holds that $(1-\epsilon)^2\epsilon - p_3 \geq 2\epsilon(1-\epsilon)^2 - p_2$ which implies $p_2 - p_3 \geq \epsilon(1-\epsilon)^2$.

(iii) Since a_1 does not want to swap with a_i , it holds that $(1-\epsilon)^2 - p_2 \geq (1-\epsilon) - p_1$ which implies $p_1 - p_2 \geq (1-\epsilon)\epsilon$.

Therefore $\epsilon - \epsilon^3 \geq p_1 - p_3 = (p_1 - p_2) + (p_2 - p_3) \geq (1-\epsilon)\epsilon + \epsilon(1-\epsilon)^2$, which is impossible for $0 < \epsilon < 1/2$. $\square$

We first characterize the displacement-CE.

Proposition 4. *An assignment is a displacement-CE assignment if and only if it is block-decomposable with all blocks of size at most 2 (with the exception that the last block can be of size 3). Any such assignment³ is a displacement-CE with price vector $(\epsilon, \epsilon^2, \dots, \epsilon^{n-1}, \epsilon^{n-1})$.*

Proof. The necessary claim is Lemma 6. We turn to show sufficiency.

³Due to identification, such assignments can be formed by non-overlapping swaps of adjacent elements from either the assignment $(a_1, \dots, a_n)$ or the assignment $(a_1, \dots, a_{n-2}, a_n, a_{n-1})$. Thus, an equilibrium assignment may end with the triplet block (a_n, a_{n-2}, a_{n-1}) .

For $n=3$, each agent solely cares about his own position and every assignment is a displacement-CE assignment with supporting price vector $(\varepsilon, \varepsilon^2, \varepsilon^2)$.

For $n \geq 4$, let $x=(a_1, \dots, a_n)$ and $p=(\varepsilon, \varepsilon^2, \dots, \varepsilon^{n-1}, \varepsilon^{n-1})$. When $i < n$, agent a_i 's payoff is $(1-\varepsilon)\varepsilon^{i-1}-\varepsilon^i$. If he moves to $k < i$, then his payoff remains the same:

$$U_{x_i}(x^{i \rightarrow k}) - p_k = \underbrace{[\varepsilon^{i-k}(1-\varepsilon) + (1-\varepsilon^{i-k})\varepsilon]}_{\text{Probability of defeating } w_{k+1}} \cdot \underbrace{\varepsilon^{k-1} - \varepsilon^k}_{\text{Price of position } k} = (1-\varepsilon)\varepsilon^{i-1} - \varepsilon^i = U_{x_i}(x) - p_i$$

Probability of defeating $x_1, \dots, x_{k-1}$

If he moves to a position k where $i < k < n$ (moves to position n do not need to be considered because the last two positions are identified), his payoff is

$$U_{x_i}(x^{i \rightarrow k}) - p_k = (1-\varepsilon)^{k-i+1} \varepsilon^{i-1} - \varepsilon^k = \varepsilon^{i-1} ((1-\varepsilon)^{k-i+1} - \varepsilon^{k-i+1}) \leq U_{x_i}(x) - p_i$$

since $(1-\varepsilon) - \varepsilon \geq (1-\varepsilon)^{k-i+1} - \varepsilon^{k-i+1}$ because $k \geq i+1$ and $0 < \varepsilon < 1/2$.

Consider now an assignment y which is formed by non-overlapping swaps of adjacent positions from x . If $y_i = a_i$, then by Lemma 1, $U_{y_i}(y) - p_i = U_{y_i}(x) - p_i$. If $y_i = a_{i+1}$ (or a_{i-1}), then again $U_{y_i}(y) - p_i = U_{y_i}(x) - p_{i+1}$ (respectively, $U_{y_i}(x) - p_{i-1}$) by the above inequality which holds with equality when $k = i+1$. Furthermore, again by Lemma 1, $U_{y_i}(y^{i \rightarrow j}) = U_{y_i}(x^{i \rightarrow j})$ where $x_j = y_i$, because the relative positions of agents stronger than him is the same in both assignments. Thus, a_i cannot gain from a displacement in (y, p) since he could not gain from a displacement from (x, p) . $\square$

4.3 Swap-CE

Proposition 5. *The swap-CE assignments are $(a_1, a_2, \dots, a_n)$ and $(a_2, a_1, a_3, \dots, a_n)$.⁴ They are supported, up to the addition of a constant to all prices, by $(\varepsilon, \varepsilon^2, \dots, \varepsilon^{n-1}, \varepsilon^{n-1})$.*

Proof. For $n=3$, each agent solely cares about his own position and every assignment is a swap-CE assignment with supporting price vector $(\varepsilon, \varepsilon^2, \varepsilon^2)$. From now on, suppose that $n \geq 4$. By Lemma 6 every swap-CE assignment is block-decomposable with all blocks of size 1 or 2 (with the possible exception that the last block can be of size 3).

We show now that the only possible doubleton block is $B_1 = (a_2, a_1)$.

⁴For $n=3$, all three assignments are swap-CE assignments.

Case 1: All cases except that the sequence begins with $(a_2, a_1, a_4, a_3, \dots)$ or ends with $(\dots, a_n, a_{n-2}, a_{n-1})$.

For a contradiction, let $r < n-1$ be the beginning of the first doubleton which is not (a_2, a_1) . Then $x_{r-1} \succ x_{r+1} \succ x_r$, agents in positions $r+2$ onward are weaker than x_{r+1} and agents in positions up to $r-2$ are stronger than x_{r-1} . The following are necessary inequalities:

(i) Since x_{r-1} does not want to swap with x_r we have

$$\varepsilon^{r-2}(1-\varepsilon) - p_{r-1} \geq \varepsilon^{r-2}(1-\varepsilon)^2 - p_r$$

and thus $\varepsilon^{r-1} - \varepsilon^r \geq p_{r-1} - p_r$.

(ii) Since x_r does not want to swap with x_{r+1} ,

$$q\varepsilon[(1-\varepsilon)\varepsilon^{r-1}] + (1-q\varepsilon)\varepsilon^r - p_r \geq q(1-\varepsilon)\varepsilon^r + (1-q)\varepsilon^{r+1} - p_{r+1}$$

where $q = \Pr(w_{r+2} \prec x_r)$ and thus $\varepsilon^r - \varepsilon^{r+1} \geq p_r - p_{r+1}$.

(iii) Since x_{r+1} does not want to swap with x_{r-1} :

$$(1-\varepsilon)^2\varepsilon^{r-1} - p_{r+1} \geq (1-\varepsilon)^2\varepsilon^{r-1} + (1-(1-\varepsilon)^2)(1-\varepsilon)\varepsilon^{r-2} - p_{r-1}$$

and thus $p_{r-1} - p_{r+1} \geq (1-(1-\varepsilon)^2)(1-\varepsilon)\varepsilon^{r-2} = (2-\varepsilon)(1-\varepsilon)\varepsilon^{r-1}$.

Thus, $\varepsilon^{r-1} - \varepsilon^{r+1} \geq p_{r-1} - p_{r+1} \geq \varepsilon^{r-1}(1-\varepsilon)(2-\varepsilon)$, which is impossible when $\varepsilon < 1/2$.

Case 2: The sequence begins with $(a_2, a_1, a_4, a_3, \dots)$ or ends with (a_n, a_{n-2}, a_{n-1}) .

Since a_2 does not want to swap with a_4 , $p_1 - p_3 \leq \varepsilon - \varepsilon^3$. Since a_4 does not want to swap with a_3 we get that $p_3 - p_4 \leq \varepsilon^3(1-\varepsilon)$. Since a_3 does not want to swap with a_2 we have that $p_1 - p_4 \geq \varepsilon(1-\varepsilon)(1+2\varepsilon-\varepsilon^2)$. The three inequalities are contradictory.

A similar calculation applies to the ending case with either positions $n-3, n-2, n-1$ (if preceded by a singleton block) replacing positions 1,3,4. If the ending is preceded by a doubleton, then the assignment is $(a_2, a_1, a_5, a_3, a_4)$ and positions 1,3,4 suffice.

We conclude that a swap-CE assignment must be $(a_1, a_2, \dots, a_n)$ or $(a_2, a_1, a_3, \dots, a_n)$. For both assignments, the adjacent no-swap conditions imply $p_i - p_{i+1} = \varepsilon^i(1-\varepsilon)$ for $i \leq n-2$, and since the last two positions are payoff-equivalent, $p_{n-1} = p_n$. Thus, up to the addition of a constant to all prices $p = (\varepsilon, \varepsilon^2, \dots, \varepsilon^{n-1}, \varepsilon^{n-1})$.

Conversely, it is easy to see that both assignments with these prices are swap-CE. $\square$

Thus, the sequential-elimination model has the property (which does not hold in general) that every swap-CE is also a displacement-CE.

5. Serial Dictatorship

In this section, a *situation* is a triple $(O, \{S_a\}_{a \in A}, \succ)$, where O is a set of n objects, $S_a \subseteq O$ is the set of objects that agent a finds satisfactory, and $\succ$ is a strict tie-breaking ordering on O . Agent a 's utility from object o is $v_a(o) = 1$ if $o \in S_a$ and 0 if $o \notin S_a$.

An assignment $(x_1, \dots, x_n)$ specifies a *pick order*. Starting with x_1 , the agents choose one at a time from the objects that remain after the agents ahead of them have chosen. Each agent chooses the $\succ$ -highest satisfactory object if one is available, and otherwise the $\succ$ -highest remaining object. An allocation of objects to agents is a *matching*.

Note that all agents use the same tie-breaking order, both among satisfactory objects and among unsatisfactory objects. One interpretation is that the tie-breaking order represents prestige and agents have lexicographic preferences: each agent's first consideration is his own payoff (namely, his utility minus any price that he pays) and his second is prestige.

The externalities in this model are due to the fact that an agent's evaluation of a position depends upon the sequence of agents ahead of him, because they may pick all the objects he deems satisfactory before he gets to choose.

Since different agents find different objects satisfactory, it may or may not be possible for all agents to receive a satisfactory object.⁵ Any all-satisfactory matching can be induced by ordering the agents according to the tie-breaking ranking of their matched objects. We call a matching *efficient* if it maximizes the number of agents who receive a satisfactory object.⁶

Note that unlike the first two sections, in the serial dictatorship setting, the two CE concepts are not nested. The difference between the two concepts is in the desirability of moves to the right. That is, when an agent x_i moves left from position i to position $j < i$, he does not care whether x_j moves to position $j+1$ or position i . However, when an agent x_i moves right from position i to position $j > i$, the value of the move if x_j moves to position $j-1$ or position i since x_i 's choice set in position j can differ based upon where x_j moved.

⁵Hall's Marriage Theorem (Hall, 1935) provides an exact condition for when all agents can be satisfied: for every set of agents, the number of objects that at least one of them finds satisfactory is at least as large as the number of agents in the set.

⁶It can be shown that any efficient matching is the outcome of some pick order by a similar construction to that of Proposition 6.

Recall that every swap-CE is pairwise stable (Section 2). In the previous two sections, every displacement-CE was also pairwise stable, but this is not true in the current setting: Let $S_{a_1}=\{o_3\}$, $S_{a_2}=\{o_1\}$, $S_{a_3}=\{o_1, o_3\}$, with tie-breaking order $o_1 \succ o_2 \succ o_3$. The pick order (a_1, a_2, a_3) induces the matching (o_3, o_1, o_2) and together with the price vector $(1, 1, 0)$ is a displacement-CE, but it is not pairwise stable: a_1 and a_3 can profitably swap. A pairwise-stable pick order in this example is (a_2, a_1, a_3) .

5.1 Existence

The existence of the three solution concepts discussed in this paper is not guaranteed in the current setting. The following simple $n=4$ example shows that both CE concepts can fail to exist. Finding an example with no pairwise-stable pick orders is difficult. ChatGPT claims one with $n=18$. We omit it because it offers no insight into the model.

Example A. *A swap-CE and a displacement-CE can fail to exist.*

There are two agents, a_1 and a_2 of type A , two agents, b_1 and b_2 of type B , and $O=\{o_1, o_2, o_3, o_4\}$. Their satisfaction sets are $S_{a_1}=S_{a_2}=\{o_2\}$ and $S_{b_1}=S_{b_2}=\{o_3\}$. The tie-breaking order is $o_1 \succ o_2 \succ o_3 \succ o_4$.

Up to relabeling agents within each type, we consider the three pick orders that start with a type A agent (the pick orders in which a type B agent picks first are similar):

For $x=(a_1, a_2, b_1, b_2)$, the induced matching is $z=(o_2, o_1, o_3, o_4)$. Agents a_2 and b_2 are unsatisfied and swapping them satisfies b_2 , so x is not pairwise stable and hence not a swap-CE. If there were a displacement-CE (x, p) , then p_2 must be 0, and b_2 could move to position 2 and become satisfied.

For $x=(a_1, b_1, b_2, a_2)$, agent a_1 is satisfied and b_2 is not. Swapping them leaves a_1 satisfied and satisfies b_2 , so x is not pairwise stable and thus not a swap-CE pick order. Suppose (x, p) is a displacement-CE. The two unsatisfied agents' positions must have prices $p_3=p_4=0$. Moving from position 1 to 3 leaves a_1 satisfied, so $p_1 \leq p_3=0$. Then any unsatisfied agent could move to position 1 and become satisfied, a contradiction.

Finally, consider $x=(a_1, b_1, a_2, b_2)$ which is pairwise-stable, but not a CE pick order. Suppose (x, p) is a swap-CE or a displacement-CE. The prices for the unsatisfied positions must be $p_3=p_4=0$. If b_1 swaps with a_2 or displaces a_2 , then b_1 remains satisfied, so $0=p_3 \geq p_2$, but b_2 could profitably swap or displace b_1 , a contradiction. $\square$

Although a pairwise-stable pick order need not exist in general, its existence is guaranteed for certain classes of situations. The first is when every satisfaction set contains at most one object; in the second, the satisfaction sets are nested. In both classes, a pairwise-stable pick order always exists, while a swap-CE and a displacement-CE may not exist (see Examples A and C).

Example B. *Pairwise-stable pick orders exist when satisfaction sets are singletons or empty.*

Suppose that $|S_a| \leq 1$ for every agent a . Partition the objects into O_2 , those desired by more than one agent; O_1 , those desired by exactly one agent; and O_0 , those desired by no agent. For each object in O_2 and O_1 , select one agent who desires it, and denote the sets of selected agents as A_2 and A_1 , respectively. Let $U = A \setminus (A_1 \cup A_2)$.

Construct the pick order as follows: place the agents in A_2 first, followed by those in A_1 , and then those in U . Within A_2 and U , order agents according to the $\succ$ -ranking of their desired objects and place agents with empty satisfaction sets at the end of U . Agents in A_1 can be ranked arbitrarily. With this pick order, every agent in $A_2 \cup A_1$ obtains his desired object, whereas every agent in U is unsatisfied.

Consider an agent $u \in U$ with $S_u = \{g\}$ (and thus $g \in O_2$). For him to become satisfied by swapping with another agent v , it must be that $v \in A_2$ and that v is positioned weakly ahead of the agent who takes g . Let $S_v = \{h\}$. If $g = h$, then u takes g and v becomes unsatisfied. If $g \neq h$, it must be that $h \succ g$ (because A_2 was ordered by the $\succ$ -ranking of the agents' desired objects). Since $h \in O_2$, another agent in U desires h and precedes u in U (because U is also ordered by the $\succ$ -ranking of the agents' desired objects). Therefore, h is taken before v gets to choose and v is again unsatisfied. Therefore, no swap increases the two agents' total payoff, so the pick order is pairwise stable. $\square$

Example C. *Nested satisfaction sets.*

When the satisfaction sets are nested, the CE concepts may not exist.⁷ We now show that a pairwise-stable pick order always exists.

Assume that $\emptyset \subsetneq S_{a_1} \subseteq \dots \subseteq S_{a_n}$. Go through the agents in the order $a_1, \dots, a_n$ while maintaining two lists H and U , initially empty. When considering a_i , append him to H if

⁷For example: $n=4$, $S_1=S_2=\{o_2\}$, $S_3=S_4=\{o_1, o_2, o_4\}$, and $o_1 \succ o_2 \succ o_3 \succ o_4$.

$|S_{a_i}| > |H|$; otherwise, append him to U . After considering all agents, construct the pick order x by placing the members of H first, in the order in which they joined H , followed by the members of U in any order.

Under x , every agent in H is satisfied since his satisfaction set is larger than the number of agents in front of him. Furthermore, under x , every agent $u \in U$ is unsatisfied: Put $m = |S_u|$. At least m agents had already joined H when u was considered, all of whose satisfaction sets are contained in S_u . Thus, they take all objects in S_u before u has an opportunity to pick.

A swap between two agents in H cannot increase their total payoff, since both are satisfied. Nor can a swap between two agents in U satisfy either agent, since the agents in H exhaust the satisfaction sets of every agent in U . Finally, consider $u \in U$ and $v \in H$. Let $m = |S_u|$. Agent u can become satisfied only if v is among the first m members of H . In that case, after the swap, u together with the other $m - 1$ such agents exhausts S_u . Since $S_v \subseteq S_u$, v is then unsatisfied. Hence no swap increases their total payoff. $\square$

We have seen that the solution concepts do not necessarily exist when we take the tie-breaking rule as *given*. The next proposition establishes a version of the Second Welfare Theorem for a suitable tie-breaking rule, every efficient matching can be induced by a pick order and a price vector that is simultaneously a swap-CE and a displacement-CE.

Proposition 6. *Consider a pair $(O, \{S_a\}_{a \in A})$ and any efficient matching $z = \{z_a\}_{a \in A}$. There is a tie-breaking rule $\succ$, a pick order x , and a price vector p such that (x, p) is both a displacement-CE and a swap-CE and induces z (thus, x is also a pairwise-stable pick order).*

Proof. Let H and U be the sets of happy (satisfied) and unhappy (unsatisfied) agents in z . If U is empty, then assign any tie-breaking rule $\succ$, order the agents according to the $\succ$ -ranking of their matched alternative, and set all prices equal to 0. Since all agents are satisfied and pay 0, there is no incentive to move.

From now on, assume that U is non-empty. Draw an arrow $b \rightarrow a$ whenever $z_b \in S_a$ (that is b 's object, z_b , satisfies a), and let R be the set of agents reachable by a directed path $\rightarrow$ starting at any agent in U , including U itself. Set $H_1 = H \setminus R$ and $H_2 = H \cap R$. For any set of agents B , let $Z(B) = \{z_b : b \in B\}$. We will now show that:

$$S_a \cap Z(R) = \emptyset \quad \text{for every } a \in H_1 \cup U. \quad (*)$$

If (*) fails for $a \in H_1$ then a would be in R and $a \notin H_1$, a contradiction. If (*) fails for $a \in U$, there is a path from U to b , a holder of an object desired by a . Reassigning the objects along the path and then assigning z_b to a would increase the number of satisfied agents, contrary to the efficiency of z .

Choose a pick order x with the blocks H_1, U, H_2 , in that order, set the tie-breaking rule $z_{x_1} \succ \dots \succ z_{x_n}$, and define $p_i = 1$ if $i \leq |H_1|$ and $p_i = 0$ otherwise. The pick order induces z . Each satisfied agent takes his assigned object, which is highest-ranked among those remaining. By (*), an agent in U finds no satisfactory object when called and therefore also takes his assigned object.

Agents in H_2 already obtain the maximum payoff, 1. An agent in U cannot gain by moving to a price 1 position; after a move within the price-0 block, all objects in $Z(H_1)$ are still taken before he chooses, so (*) rules out a gain there as well.

Finally, an agent $a \in H_1$ obtains payoff 0 and could gain only by a swap or a displacement to a price-0 position. After the move all $|H_1| - 1$ other agents in H_1 and the first agent in U choose before a . By (*) and the tie-breaking order, each of these $|H_1|$ agents takes an object in $Z(H_1)$. Thus $Z(H_1)$ is exhausted before a chooses; (*) rules out a satisfactory choice for him from the remaining objects. His payoff is still 0. $\square$

5.2 Efficiency

We first show a version of the First Welfare Theorem: every displacement-CE matching is efficient. We then give an example of an inefficient swap-CE matching, and hence of an inefficient pairwise-stable pick order.

Proposition 7. *Every displacement-CE matching is efficient.*

Proof. Let (x, p) be a displacement-CE which induces the matching $z = (z_a)_{a \in A}$. Since earlier picks have access to more options, it must be that prices are weakly decreasing, $p_1 \geq p_2 \geq \dots \geq p_n = 0$. We need the following lemma.

Lemma 7. *An agent who pays a positive price for his position in a displacement-CE finds every object assigned to an agent in a zero-price position unsatisfactory.*

Proof. Let m be the first position for which $p_m = 0$. Thus, $p_1, \dots, p_{m-1} > 0$ and $p_m = \dots = p_n = 0$. Every agent in a positive-price position must receive a satisfactory object. Other-

wise, his payoff would be negative, whereas moving to the last position would give him a nonnegative payoff. Note that there may also be satisfied agents who pay 0.

Suppose, contrary to the claim, that there is an agent $a=x_i$ with $i < m$ (who pays a positive price) and an agent $b=x_j$ with $j \geq m$ (who pays 0) such that $z_b \in S_a$. Consider a move of a to position m . When a leaves position i , the object z_a becomes available. Each of the agents in positions $i+1, \dots, m$ moves one position earlier, and faces his original menu together with one additional object. Hence, by the fixed tie-breaking rule, each such agent either retains his original object or picks the additional object. In the latter case, his original object becomes the new displaced object.

If $j > m$, none of the shifted agents takes z_b . Then, z_b remains available to a if he moves to position m and picks a satisfactory object for the price 0, which is a profitable move. Suppose $j = m$. Let o be the displaced object available at position m after agent a moves there. If $o \in S_a$, then a remains satisfied when he moves to position m and pays 0, which is a profitable move. If $o \notin S_a$, then $o = z_c$ for some agent c who occupies a position between i and m . In this case, c could profitably move to position m and obtain o as nobody between his position and m would pick it (because they did not pick it when it was available to them in the displacement chain). $\square$

Back to the proof of the proposition: Assume by contradiction that there is a matching y with more agents satisfied than in z . Then, there is an agent a who is unsatisfied in z and is satisfied in y . Let $y_a = z_b$. Agent a pays 0 in the equilibrium and agent b is ahead of agent a in x and pays 1 (otherwise, a would move to b 's position). In y , agent b is matched with an unsatisfactory object unless, by Lemma 7, y_b is held in z by another agent c who pays a positive amount for his position and thus must be satisfied in z . Then repeat the same analysis with c to construct a sequence of agents until it reaches a previously satisfied agent who is matched in y with an unsatisfactory object. Thus any move of an agent from an unsatisfactory object in z to a satisfactory object in y is coupled with an agent moving in the other direction. A contradiction. $\square$

Example D. *Inefficient Swap-CE.*

Let $O = \{a, b, c, d, e\}$, with tie-breaking order $a \succ b \succ c \succ d \succ e$, and let

$$S_1 = \{c, d\}, \quad S_2 = \{a\}, \quad S_3 = \{c, e\}, \quad S_4 = \{b\}, \quad S_5 = \{a, e\}.$$

The pick order $x = (1, 2, 3, 4, 5)$ produces the matching (c, a, e, b, d) in which agents 1, ..., 4 are satisfied, but agent 5 is not. This matching is inefficient since the matching

(d, a, c, b, e) satisfies all agents. The pick order is a swap-CE with prices $p=(1,1,1,1,0)$. The only potential profitable swaps move a satisfied agent i to position 5, but in any such swap i will not obtain a satisfactory object (note in particular that if 4 swaps to the last position, agent 5 takes b although he is not satisfied by it). $\square$

6. Discussion

In standard competitive equilibrium settings, with or without externalities, each agent maximizes his own payoff, and either does not care about the actions of others, or if he does care, takes them as given. In this paper, we considered situations in which evaluating the consequences of one agent's move necessarily requires describing its effect on the actions of other agents, since those actions affect the moving agent's interests. This led us to look for equilibrium notions that specify what happens to everyone else when an agent moves.

We conducted our research in a context where agents compete for positions in a sequence. We considered the notion of swap-CE in which an agent swaps positions with another and the notion of displacement-CE where the moving agent displaces his target position while keeping the relative order of all other agents intact.

The concepts were operationalized in three different settings. In the first, agents are lined up to be interviewed for a single opening. Both equilibrium notions, as well as pairwise stability, were shown to lead to near search-minimizing orders (namely the more promising a candidate is, the earlier he is in the line).

In the second, agents are sequentially eliminated through pairwise comparisons from right to left. Agents differ in strength, and upsets occur with some positive probability. Both equilibrium notions result that the agents are roughly ordered from the strongest to the weakest.

The third setting is a serial dictatorship problem in which participants pick objects sequentially. But prices are for the positions, not for the objects themselves. This setting is a bit different and equilibria may not exist. Nevertheless, we establish versions of both welfare theorems. The first states that: displacement-CE are always efficient, whereas swap-CE and pairwise stability need not be. The second state that: every efficient matching induced by some displacement/swap-CE with some tie-breaking rule.

Overall, the paper is motivated by the observation that in some markets with externalities, a definition of competitive equilibrium requires specifying how the choices of other agents are affected by an individual's deviation. Our analysis demonstrates how this specification can affect the existence, nature, and efficiency of equilibrium assignments.

7. References

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