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Choice problems with a 'reference' point

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Abstract

In many decision scenarios, one has to choose an element from a set S given some reference point e. For the case where S is a subset of a Euclidean space, we axiomatize the choice method that selects the point in S that is closest to e.

Keywords: Choice; Problems; Locations; Theories

1. Introduction

Many choice problems have the structure (S,e), where S is a set of feasible alternatives in some ambient space X and e is a reference point in X. A choice function assigns to each choice problem (S,e) within its domain a unique element in S.

We have two typical scenarios in mind.

1. The set X contains all potential locations of a project. The previously agreed upon location is e, and the feasible locations are points in S, which may or may not include e. A choice method has to specify one point in S as the new location.
2. The set X includes all possible theories (point-beliefs) about the world. The current accepted theory is e. A new discovery, however, indicates that the world is actually in the set S. A choice method has to specify a theory in S given the previous theory e, and the new information designated by S. This interpretation brings our problem close to that of 'belief updating' as formulated by Lewis (1973); Stalnaker (1968).

Note the difference between the above scenarios and the scenario behind the Nash

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bargaining problem, which has also the structure (S,e). In its common interpretation, the set S is taken to be the set of all feasible utility vectors that various parties can agree upon and the point e is the disagreement point. The bargaining problem can also be thought of as a single decision-maker problem, where e is the status quo option, S is the set of alternatives, and the different 'bargainers' stand for various considerations that are involved in making the decision. In its two interpretations, the bargaining problem is evolved when there is a conflict between interests (bargainers or internal motives) specified by the model. In contrast, we do not specify such interests here. Also, while e always belongs to S in the Nash bargaining problem (S,e), this is not required in our model.

We require that a choice method assigns to each problem (S,e) an element in S and we allow problems in which $e \in S$ as well as problems in which $e \notin S$. Thus, we exclude the possibility that $f(S,e)=e$ for all choice problems. In other words, we restrict the discussion to cases in which staying at the reference point e is not always possible. Notice that this does not make our model more general, as the extension of the domain of the choice method makes the axioms which we will employ more demanding.

Our main interest in this paper is to axiomatize the 'minimal distance' choice function that selects the point in S which is the closest to e. The choice function has a natural meaning especially in the context of belief updating.

We first show (in an Euclidean space X) that the axiom of independence of irrelevant alternatives à la Nash [which requires that if x is the choice of the problem (S,e) and x is in $T \subset S$, then x should also be the solution for (T,e)] and a version of the symmetry axiom, stronger than Nash's symmetry axiom, characterize the method that chooses the point in S that is closest to S.

The minimal distance choice function does not satisfy a path independence axiom. In the belief-updating scenario, this axiom requires that if information is brought up sequentially, the order by which it is raised be immaterial. Our second result is that no choice function satisfies the axiom if X is of dimension higher than one.

2. The minimal distance choice

In this section, we characterize the method that always selects the element in S that is closest to e. Of course, such a method depends on the existence of a distance function. We, therefore, start by a space X that, for simplicity, is taken to be the Euclidean space (it is straightforward to extend the analysis to any Hilbert space) with the distance function $d(x,y)$. A choice problem (S,e) is taken to be a pair where S is a closed convex subset of X and $e \in X$. In this setting, $\arg\min_{x \in S} d(x,e)$, the function that attaches to (S,e) the point in S that is closest to e, is well defined.

For the axiomatization we need some basic definitions. A line $\langle a,\alpha \rangle$ is the set of all points of the form $a+t\alpha$ for some real t. We say that a set S is symmetric relative to $\langle a,\alpha \rangle$ if, for every β satisfying $\alpha^{\mathrm{T}}\beta=0$ (namely β is orthogonal to the direction α), $a+t\alpha+\beta \in S$ implies $a+t\alpha-\beta \in S$.

Axiom 1. *(Symmetry or SYM). If S is symmetric relative to a line $\langle e,\alpha \rangle$, then $f(S,e) \in \langle e,\alpha \rangle$.*

The other axiom we employ is the standard Nash independence of irrelevant alternatives.

Axiom 2. *(Independence of Irrelevant Alternatives or IIA). If $T \subset S$ and $f(S,e) \in T$, then $f(T,e) = f(S,e)$.*

Proposition 2.1. *The only choice function that satisfies SYM and IIA is the minimal distance choice function*

$$f(S,e) = \arg\min_{x \in S} d(x,e).$$

Proof. It is straightforward to verify that the minimal distance choice function indeed satisfies SYM and IIA.

On the other hand, assume that a choice function f satisfies SYM and IIA. We want to show that $f(S,e) = \arg\min_{x \in S} d(x,e)$ for every (S,e).

Case 1. $e \in S$.

If $S = X$, then $f(X,e) = e$ since X is symmetric to every line passing through e. Then, by IIA, $f(S,e) = e$ for every (S,e) with $e \in S$.

Case 2. $e \notin S$.

Assume first that S is a half space H. Suppose $f(H,e) = z \neq y = \arg\min_{x \in H} d(x,e)$.

If z is on the boundary of H, then the point $z' = z + 2(y - z)$, which satisfies $y = (z + z')/2$, also belongs to H. Then $f([z,z'],e) = z$ by IIA, but $f([z,z'],e) = y$ since $[z,z']$ is symmetric to the line $\langle e, e - y \rangle$ and y is the only point in the intersection of $[z,z']$ and $\langle e, y - e \rangle$; thus, we see a contradiction.

If z is in the interior of H, then find another point $z' \in H$ such that $d(z',e) = d(z,e)$. By SYM, $f([z,z'],e) = (z + z')/2$. But by IIA, $f([z,z'],e) = z$, again a contradiction.

Finally, let S be any closed convex set. Find the hyperplane h that supports S at y with $y - e$ as the normal vector, and the half space H determined by h to which e does not belong. Then, according to the proof in Case 1, $f(H,e) = y$, and by IIA, $f(S,e) = y$. ■

It is obvious that SYM and IIA are independent: A choice function that chooses the minimal distance point for every symmetric choice problem yet arbitrary otherwise satisfies SYM but not IIA; a choice function that always chooses the maximal point of some strictly concave function satisfies IIA but not SYM.

Note that in Nash's axiomatization Nash (1950), the symmetry axiom applies to problems that are symmetric to the main diagonal only. The uniqueness of the Nash bargaining solution is achieved by utilizing the axiom of invariance to positive affine transformation. In contrast, we do not employ any invariance axiom, but we do impose a stronger symmetry axiom.[1]

In an early article, Yu (1973) considered (but has not characterized) the 'minimal

[1] It is true that our symmetry axiom can be derived from the conjunction of Nash's symmetry axiom and the axiom of invariance with respect to rotations. But the conjunction of both would generally be stronger than our symmetry axiom (in the absence of IIA).

distance' choice function for one type of decision problem. More recently, Conley et al. (1994) considered a characterization of this choice function. A more general form of this problem is the 'bargaining problem with claims' (Chun and Thomson (1992)) with the structure of a triple (S,e,c), with the interpretation that S is the set of feasible utility vectors, $e \in S$ is the disagreement point, and $c \notin S$ is the vector of claims that cannot be fulfilled. This model emphasizes the utility interpretation of choices. As a consequence, Pareto optimality is always imposed as an axiom regarding solutions for this problem. Our model, however, does not have the utility interpretation and Pareto optimality plays no role in our work.

3. Path independence

Let us now consider the belief updating interpretation of our model. Recall that Bayesian updating, a widely adopted method of belief revision, satisfies the property that the order by which information is received does not matter. It is natural to ask if there is any method in our framework that may satisfy this property. Formally, consider the following axiom.

Axiom 3. *(Path Independence or PI) For all e, S, and T with* $S \cap T \neq \emptyset$, $f(S \cap T,e) = f(S, f(T,e))$.

For the 'minimal distance' method, it is possible that the order by which the information is received does matter, i.e, it does not satisfy PI. For example, having $X = R^2$, take S to be the segment connecting (1,0) and (n,n), T the segment connecting (0,1) and (n,n), and $e = (0,0)$. Then $f(S \cap T,e) = (n,n)$ whereas $f(S, f(T,e))$ converges with n to (1,0). Notice that the minimal distance method satisfies PI for $X = R^1$.

Proposition 3.1 *If* $X = R^k$ *with* $k \geq 2$, *then no choice function satisfies PI.*

Proof. Suppose that a choice function f satisfies PI. For any (S,e) with $e \in S$, by PI, $e = f(\{e\},e) = f(S \cap \{e\},e) = f(S, f(\{e\},e)) = f(S,e)$.

Take any three distinct elements, a, b and e that are not on the same straight line. This is possible since X is more than one dimensional. Let $f([a,b],e) = c$. Without loss of generality, assume that $c \neq a$. Using PI, we have $a = f(\{a\},e) = f([a,b] \cap [a,e],e) = f([a,b], f([a,e],e)) = f([a,b],e) = c$, which is a contradiction. ■

If we expand the scope of choice functions by allowing $f(S,e) \in S \cup \{e\}$, we can show that the only choice function satisfying PI is the one that always chooses the reference point e for every problem (S,e). The proof of Proposition 2 carries though. The first paragraph shows that $e = f(S,e)$ for any (S,e) with $e \in S$. The second shows that for $f([a,b],e) = e$ for all a, b and e that are not on the same line. Now for any $a \neq e$, find b and c such that: (i) a, b, and e are not on the same line; (ii) a, c and e are not on the same line; and (iii) $[a,b] \cap [a,c] = \{a\}$. We then have $f(\{a\},e) = f([a,b] \cap [a,c],e) = f([a,b], f([a,c],e)) = f([a,b],e)) = e$. Finally, if $f(S,e) = a \neq e$, then $e = f(\{a\},e) = f(S \cap \{a\},e) = f(S, f(\{a\},e)) = f(S,e) = a$, a contradiction.

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References

Chun, Y., Thomson, W., 1992. Bargaining problems with claims. Mathematical Social Sciences 24, 19–33.

Conley, J., McLean, R., Wilkie, S., The duality of bargaining theory and multi-objective programming, and generalized choice problems, Working Paper, University of Illinois, 1994.

Lewis, D., Counterfactuals, Blackwell, 1973.

Nash, J.F., 1950. The bargaining problem. Econometrica 18, 155–162.

Stalnaker, R., A theory of conditionals. In: Rescher, N. (Ed.), Studies in Logical Theory, Blackwell, 1968.

Yu, P.L., 1973. A class of solutions for group decision problems. Management Science 19, 936–946.