JOURNAL OF ECONOMIC THEORY **70**, 32–42 (1996)
ARTICLE NO. 0074

An Extensive Game as a Guide for Solving a Normal Game*

JACOB GLAZER

Faculty of Management, Tel Aviv University

AND

ARIEL RUBINSTEIN

School of Economics, Tel-Aviv University, and
Department of Economics, Princeton University

Received November 16, 1993; revised July 12, 1995

We show that for solvable games, the calculation of the strategies which survive iterative elimination of dominated strategies in normal games is equivalent to the calculation of the backward induction outcome of some extensive game. However, whereas the normal game form does not provide information on how to carry out the elimination, the corresponding extensive game does. As a by-product, we conclude that implementation using a subgame perfect equilibrium of an extensive game with perfect information is equivalent to implementation through a solution concept which we call guided iteratively elimination of dominated strategies which requires a uniform order of elimination. *Journal of Economic Literature* Classification Number: C72.

1. INTRODUCTION

Game theory usually interprets a game form as a representation of the physical rules which govern a strategic interaction. However, one can view a game form more abstractly as a description of a systematic relationship between players' preferences and the outcome of the situation. Consider, for example, a situation which involves two players, 1 and 2. The players can go out to either of two places of entertainment, T or B, bringing with them a third (passive) party L or R. The two players have preferences over the four possible combinations of place and companion. The three presuppositions regarding the situation are:

* The first author acknowledges financial support from the Israel Institute of Business Research. The second author acknowledges partial financial support from the United States–Israel Binational Science Foundation, Grant Number 1011-341. We thank Paolo Battigiali, an associate editor, and a referee of this Journal, for their excellent comments on the first version of this paper.

0022-0531/96 $18.00

	L	R
T	a	b
B	c	d

FIGURE 1

(i) Player 2's preferences over the companion component are independent of the place of entertainment.

(ii) Player 2 decides on L or R.

(iii) Player 1 decides on T or B.

Game theory suggests two models to describe this situation. One model would describe the players as playing the game G (see Fig. 1) and the outcome determined by the solution of successive elimination of weakly dominated strategies. The other would say that the players are involved in the game Γ (see Fig. 2) and that the solution concept is one of backward induction. Both alternatives summarize all the information we possess about the situation. However, the description of the situation via an extensive game is more informative than that via a normal game form since the former provides a guide for easier calculation of the outcome for any given profile of preferences which is consistent with (i).

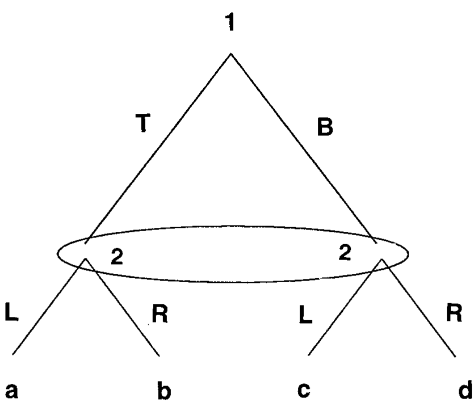

FIGURE 2

In this paper we elaborate on this idea. We begin in Section 2 by introducing the notion of a "guide" for solving normal form games through iterative elimination of dominated strategies. A guide is a sequence of instructions regarding the order of elimination. In Section 3 we establish that the information about the procedure of solving a normal form game provided by the guide is essentially identical to the additional information which is provided when the game is described in its extensive form rather than its normal form. As a by-product, we show in Section 4 that implementation by subgame perfect equilibrium (SPE) in an extensive game is equivalent to implementation through a solution concept, which we call guided iteratively undominated strategies, in a normal game which requires a uniform order of elimination.

2. Preliminaries

Let N be a set of players and C be a set of consequences. A preference profile is a vector of preferences over C, one preference for each player. In order to simplify the paper we confine our analysis to preferences which exclude indifferences between consequences.

(a) *Normal Game Form*

A *normal game form* is $G = \langle \times_{i \in N} S_i, g \rangle$, where S_i is i's strategy space and $g\colon \times_{i \in N} S_i \to C$ is the consequence function. (Without any loss of generality, assume that no strategy in S_i has the name of a subset of S_i.) A game form G accompanied by a preference profile $p = \{\geqslant_i\}_{i \in N}$ is a *normal game* denoted by $\langle G, p \rangle$. We say that the strategy $s_i \in S_i$ *dominates* the strategy $s'_i \in S_i$ if $g(s_i, s_{-i}) \geqslant_i g(s'_i, s_{-i})$ for any profile $s_{-i} \in \times_{j \neq i} S_j$. By this definition one strategy dominates the other even if $g(s_i, s_{-i}) = g(s'_i, s_{-i})$ for all s_{-i}.

(b) *Guide*

A guide for a normal form G is a list of instructions for solving games of the type $\langle G, p \rangle$. Each instruction k consists of a name of player i_k and a set A_k. The sublist of instructions for which $i_k = i$ can be thought of as a "multi-round tournament" whose participants are the strategies in S_i. The first instruction in this sublist is a set of at least 2 strategies for player i. One of these strategies will be thought of as a winner (in a sense that will be described later). The losers leave the tournament and the winner receives the name of the subset in which he won. Any element in the sublist is a subset of elements which are left in the tournament. Such an element is either a strategy in S_i which has not participated in any previous round of the tournament, or a strategy which won all previous rounds in which it participated; this strategy appears under the name of the last round in which it won. Following completion of the last round, only one strategy of player i remains a non-loser. Thus, for example, if $S_1 = \{x_1, x_2, x_3, x_4, x_5\}$, a possible sublist for player 1 is $A_1 = \{x_1, x_2\}$, $A_2 = \{x_3, x_4\}$, and $A_3 = \{A_1, A_2, x_5\}$. In the first round x_1 and x_2 are "compared." In the second round the strategies x_3 and x_4 are compared and in the final round x_5 and the winners of the previous two rounds are compared. The guide is an order in which the strategies are compared, but it does not contain the rules by which one strategy is declared a winner in any particular round.

Formally, a (finite) *guide* for G is a sequence $(i_k, A_k)_{k=1,\ldots,K}$ satisfying:

(i) For every k, $i_k \in N$.

(ii) For every k' with $i_{k'} = i$, $A_{k'}$ is a set with at least two elements where each element in the set is either a strategy in S_i or a set A_k with $i_k = i$ and $k < k'$.

(iii) Let k_i^* be the largest k with $i_k = i$. Each strategy in S_i and each set A_k with $i_k = i$ and $k < k_i^*$ is a member of a single set $A_{k'}$ with $i_{k'} = i$.

So far we have only defined the structure of the tournament and have yet to describe how a winner is selected in each round. A winner in round k is an element of A_k which dominates the other elements according to player i_k's preferences in the game in which all the losers of all players in the previous $k-1$ rounds were eliminated. A guide for G solves a game $\langle G, p\rangle$ if, when applying the guide, there is a winner in each round. Our formal definition is inductive. The guide $D = (i_k, A_k)_{k=1,\ldots,K}$ *solves the game* $G = \langle \times_{i\in N} S_i, g, p\rangle$ if

(i) there is an $a^* \in A_1$ which dominates all strategies in A_1 and

(ii) for $K > 1$, the guide $D' = (i_{k+1}, A_{k+1})_{k=1,\ldots,K-1}$ solves the game G' which is obtained from G by omitting all of i_1's strategies in A_1 and adding one new strategy called A_1 to player i_1's set of strategies so that $g'(A_1, a_{-i_1}) = g(a^*, a_{-i_1})$.

Thus, for the guide to solve the game it must be that in every stage there is a dominating strategy. Note that by the assumption of no-indifference. if there are two dominating strategies a^* and b^* then $g(a^*, a_{-i_1}) = g(b^*, a_{-i_1})$ for all a_{-i_1} and thus the definition of G' does not depend on which of these strategies is declared a winner.

Note that by condition (iii) in the definition of a guide, if D solves the game $\langle G, p\rangle$, then the game which is obtained in the last stage has one strategy for each player. The consequence attached to the surviving profile of strategies is called the *D-guided I-outcome.*

The notion of iterative elimination of dominated strategies can be stated, using our guide terminology, as follows: a consequence z survives the iterative elimination of dominated strategies, and, in short, is an I-outcome of the game $\langle G, p\rangle$, if there is some guide D, such that z is a D-guided I-outcome of $\langle G, p\rangle$.

(c) *Extensive Game Form*

A (finite) *extensive game form* is a four-tuple $\Gamma = \langle H, i, I, g\rangle$, where:

(i) H is a finite set of sequences called histories (nodes) such that the empty sequence is in H and if $(a_1, ..., a_T) \in H$ then $(a_1, ..., a_{T-1}) \in H$.

(ii) i is a function which assigns to any non-terminal history $h \in H$ a name of a player who has to move at the history h (a history $(a_1, ..., a_t)$ is non-terminal if there is an x so that $(a_1, ..., a_t, x) \in H$). The set of actions which $i(h)$ has to choose from is $A(h) = \{a | (h, a) \in H\}$.

(iii) I is a partition of the set of non-terminal histories in H such that if h and h' are in the same information set (an element of this partition) then both $i(h) = i(h')$ and $A(h) = A(h')$.

(iv) g is a function which assigns a consequence in C to every terminal history in H.

We confine ourselves to games with perfect recall. A *terminal information set* X is an information set such that for all $h \in X$ and $a \in A(h)$, the history (h, a) is terminal.

The following definition of a game solvable by backward induction is provided for completeness. Simultaneously we will define the B-outcome to be the consequence which is obtained from executing the procedure. Note that our definition rests on weak dominance at information sets.

Let $\Gamma = \langle H, i, I, g\rangle$ be an extensive game form. The game $\langle \Gamma, p\rangle$ is solvable by backward induction if either:

(i) the set of histories in Γ consists of only one history (in this case it can be said that the attached consequence is the B-outcome of the game) or

(ii) Γ includes at least one terminal information set and

(a) for any terminal information set X and any $h \in X$ there is an action $a^* \in A(h)$ such that for any $a' \in A(h)$ we have $g(h, a^*) \geqslant_{i(h)} g(h, a')$,

(b) the game $\langle \Gamma', p\rangle$ is solvable by backward induction where Γ' is obtained from Γ by deleting the histories which follow X and assigning the consequence $g(h, a^*)$ to any $h \in X$.

(Formally, $H' = H - \{(h, a) \mid h \in X$ and $a \in A(X)\}$, $i'(h) = i(h)$ for any $h \in H'$, $I' = I - \{X\}$ and $g'(h) = g(h, a^*)$ for any $h \in X$ and $g'(h) = g(h)$ for any other terminal history.) The B-outcome of $\langle \Gamma, p \rangle$ is the B-outcome of the game $\langle \Gamma', p \rangle$.

Note that the game form Γ in the above definition can include information sets which are not singletons. It is required that for any such information set there is an action for the player who moves at this point which is better than any other action available at this information set regardless of which history led to it. Therefore, if a game $\langle \Gamma, p \rangle$ is solvable by backward induction then the B-outcome is the unique subgame perfect equilibrium outcome of the game with perfect information which is derived from $\langle \Gamma, p \rangle$ by splitting all information sets into singletons.

(d) *A Normal Form of an Extensive Game Form*

Let Γ be an extensive game form. A *plan of action* for player i is any function s_i which has the property that it assigns a unique action only to those information sets that can be reached by s_i (the information set X is reached if there is at least one $h = (a_1, ..., a_T) \in X$ so that for every subhistory $h' = (a_1, ..., a_t)$ with $i(h') = i$, $s_i(h') = a_{t+1}$). The notion of a plan of action differs from the notion of a strategy in an extensive game in that it is not defined for information sets that can never be reached given the strategy. Define the *reduced normal form of Γ* to be the normal game form $G(\Gamma) = \langle \times_{i \in N} S_i, g \rangle$, where S_i is the set of all player i's plans of action and $g((s_i)_{i \in N})$ is the consequence reached in Γ if every player i adopts the plan of action s_i.

3. On the Equivalence between a Normal Game Form with a Guide and an Extensive Game Form

In the previous section we distinguished between an I-outcome and a D-guided I-outcome. By stating that z is an I-outcome, no information is given as to the order of elimination which leads to the observation that z is an I-outcome. On the other hand by stating that z is a D-guided I-outcome not only do we reveal that it is an I-outcome but also that it is an outcome of elimination carried out in the order described by the particular guide D. In this section we argue that an extensive game can be viewed as equivalent to a guide, and thus conclude that calculating the subgame perfect equilibrium outcome in an extensive game is simpler than calculating the outcome of an iterative elimination of dominated strategies in a normal game.

The main result of the paper is the following.

PROPOSITION 1. *For every normal game form G and a guide D there is an extensive game form Γ (independent of any preference profile) such that the normal game form of Γ is G and for all p:*

(a) *The guide D solves the normal game $\langle G, p \rangle$ iff the extensive game $\langle \Gamma, p \rangle$ is solvable by backward induction.*

(b) *A consequence z is a D-guided I-outcome of $\langle G, p \rangle$ iff it is a B-outcome of $\langle \Gamma, p \rangle$.*

Furthermore, there is a game with perfect information Γ^* so that for all p, the B-outcome of $\langle \Gamma, p \rangle$ is the same as the subgame perfect equilibrium outcome of $\langle \Gamma^*, p \rangle$.

Proof. Let $G = \langle \times_{i \in N} S_i, g \rangle$ be a game form and $D = (i_k, A_k)_{k=1,\ldots,K}$ be a guide. We construct the extensive game form so that the calculations of the I-outcome using the guide from the beginning to the end are equivalent to the calculations of the B-outcome in the extensive game starting from the end and going backward. The construction is done inductively starting from the initial history and using the information contained in the last element of the guide.

As an initial step, assign the history ϕ to i_K. Let the set $\{\phi\}$ be an information set and let $A(\phi) = A_K$. Add to the set of histories all sequences (x) of length one where $x \in A_K$.

Now assume that we have already completed t stages of the construction. For stage $t+1$ look at $k = K - t$. If it is not the largest k' so that $i_{k'} = i_k$ (that is, it is not the first time in the construction that we assign a decision to player i_k), then group into the same information set all terminal histories in the game we have constructed up to the end of stage t in which A_k was chosen. If it is the largest k' so that $i_{k'} = i_k$, then group into the same information set all terminal histories in the game we have constructed up to the end of stage t. Add to the set of histories all histories (h, x) where $x \in A_k$.

When the construction of the set of histories is complete, any terminal history h is a sequence such that for every player i there is a nested subsequence of sets which must end with a choice of a strategy, $s_i(h) \in S_i$. We attach to the terminal history h the consequence attached to $s_i(h)$ in G. It is easy to verify that Γ is a game form with perfect recall. Figure 3 illustrates the construction.

To verify that the normal form of Γ is G, note that any strategy of player i in Γ can be thought of as a choice of one strategy in S_i with the understanding that whenever he has to move he chooses an action which is a set including s_i. Furthermore, the consequence of the terminal history which results from the profile of the extensive game strategies which correspond to $(s_i)_{i \in N}$ was chosen as $g(s)$.

The game G

	r	s	t
a	O_1	O_2	O_3
b	O_4	O_5	O_6
c	O_7	O_8	O_9

The guide D

$i_1 = 1$	$A_1 = \{a, b\}$
$i_2 = 2$	$A_2 = \{r, s\}$
$i_3 = 1$	$A_3 = \{A_1, c\}$
$i_4 = 2$	$A_4 = \{A_2, t\}$

The extensive game

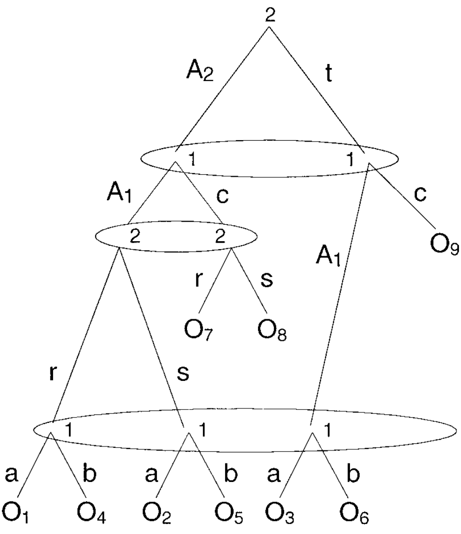

FIGURE 3

The proof of (a) and (b) follows from two observations:

(i) The first stage of calculating the backward induction in Γ and the first stage in applying D involve precisely the same comparisons. When applying D we look for a strategy $x \in A_1$ which dominates the other members of A_1; such a strategy satisfies that $g(x, a_{-i_1}) \geqslant_{i_1} g(x', a_{-i_1})$ for all $x' \in A_1$ and for all profiles a_{-i_1}. This is the calculation which is done in the first stage of the backward induction calculation in Γ. The player in the only terminal decision information set is i_1 and he has to choose an action from A_1. Since the game is with perfect recall, along each history in his information set the other players choose a single element in their strategy space. For x to be chosen, it must be that $g(x, h) \geqslant_{i_1} g(x', h)$ for all h, that is, $g(x, a_{-i_1}) \geqslant_{i_1} g(x', a_{-i_1})$ for all $x' \in A_1$.

(ii) Denote by $\Gamma(G, D)$ the extensive game form constructed from the normal game form G and the guide D. For every profile p, $\Gamma(G', D') = \Gamma'$, where G' is the normal game form obtained following the execution of the first step of the guide D, D' is the guide starting with the second instruction of D, and Γ' is the extensive game obtained by executing the first step of the backward induction procedure on Γ.

From the fact that any B-outcome of an extensive game Γ is the subgame perfect equilibrium of the extensive game Γ^* in which all information sets are singletons we conclude that there is a game form with perfect information Γ^* such that for all p, the B-outcome of $\langle \Gamma, p \rangle$ is the same as the subgame perfect equilibrium outcome of $\langle \Gamma^*, p \rangle$.

4. Implementation

It is often felt that implementation theory ignores "complexity" considerations (see [4]). A proof that a particular class of social functions is implementable frequently utilizes a game form which is messy to describe and complicated to play. It is natural to evaluate implementation devices according to their complexity in order to identify more plausible mechanisms. One component of complexity is the difficulty in calculating the outcome of the mechanism. If the calculation of the I-outcome of a normal form game involves the same comparisons as the backward induction for an extensive game, then the latter may be considered simpler in the sense that it provides the players with a guide for executing the calculation.

Let P be a set of preference profiles over C. A *social function* assigns to every profile $p \in P$ an element in C. We say that a social function f is *I-implementable* by the game form G if for all p, the I-outcome of the game (G, p) is $f(p)$. We say that a social function f is *guided-I-implementable* by the game form G and the guide D if for all p, the D-guided I-outcome of the game (G, p) is $f(p)$. In other words, the game G guided-I-implements f if there is *one* guide which solves $\langle G, p \rangle$ for all $p \in P$ and the outcome is $f(p)$. Finally, we say that f is *SPE-implementable* if there is an extensive game form with perfect information Γ so that for all p the subgame perfect equilibrium outcome of the game (Γ, p) is $f(p)$. (Actually this definition is more restrictive than the one of say [5], since only games of perfect information are admitted. It is closer to the definition of [3].)

One might conjecture that SPE-implementation is equivalent to I-implementation. This is not the case as demonstrated by the following example (suggested by the first author and Motty Perry).

EXAMPLE. Let $C=\{a, b, c, d\}$ and let $P=\{\alpha, \beta\}$ where $\alpha=(d>_1 b>_1 c>_1 a,\ b>_2 c>_2 d>_2 a)$ and $\beta=(b>_1 c>_1 d>_1 a,\ d>_2 c>_2 b>_2 a)$.

Consider the social function f: $f(\alpha)=c$ and $f(\beta)=b$. The function f is I-implementable by the normal form of the game in Fig. 1: In α, for player 1, B dominates T and, for player 2, L dominates R and the final outcome is c. In β, for player 2, R dominates L and, for player 1, T dominates B and the final outcome is b.

Notice that different orders of elimination were used in the calculation of the two profiles. In α, the elimination starts with the deletion of one of player 1's actions and in β it starts with the deletion of one of player 2's actions.

Although f is I-implementable we will now see that there is no extensive game with perfect information which SPE-implements f. If Γ is an extensive game form which SPE-implements f, then f is also SPE-implemented by a game form Γ' which is derived from Γ by the omission of all terminal histories with the consequence a (since it is the worst consequence for both players in both profiles). Let (s_1, s_2) be an SPE of (Γ', α) which results in the consequence c and let (t_1, t_2) be an SPE of (Γ', β) which results in the consequence b. It must be that in α player 1 does not gain by switching to the strategy t_1 and thus the outcome of the play (t_1, s_2) must be c. Similarly, in β, player 2 does not gain by deviating to s_2 and thus it must be that the outcome of the play (t_1, s_2) is b. This is a contradiction.

Whereas I-implementation is not equivalent to SPE-implementation, we arrive at the following equivalence:

PROPOSITION 2. *A social function f is guided-I-implementable if and only if it is SPE-implementable.*

Proof. By proposition I if f is guided-I-implementable then it is SPE-implementable. The other direction is quite simple: if we start with a game form Γ we employ the reduced normal form $G(\Gamma)$ and we construct the guide starting from the end of the extensive game.

Remark. Proposition 2 sheds new light on [1] which uses I-implementation. As It turns out, the implementation of Abreu and Matshushima is actually guided-I-implementation and this explains the fact that Glazer and Perry [2] were able to find an analogous SPE-implementation.

References

1. D. Abreu and H. Matshushima, Virtual implementation in iteratively undominated strategy, *Econometrica* **60** (1992), 993–1008.
2. J. Glazer and M. Perry, Virtual implementation in backwards induction, *Games Econ. Behav.*, in press.
3. M. Herrero and S. Strivatsava, Implementation via backward induction, *J. Econ. Theory* **56** (1992), 70–88.
4. M. Jackson, Implementation of undominated strategies: A look at bounded mechanisms, *Rev. Econ. Stud.* **59** (1992), 757–776.
5. J. Moore and R. Rappulo, Subgame perfect implementation, *Econometrica* **56** (1988), 1191–1220.