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On the Logic of "Agreeing to Disagree" Type Results*

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The analysis of the "agreeing to disagree" type results is unified by considering functions which assign to each set of states of nature the value "True" or "False". We identify properties of such functions, being preserved under union, under disjoint union and under difference. The property of being preserved by disjoint union is used to generalize Aumann's, Milgrom and Stokey's and other results. Another proposition refers to all of these properties and implies Samet's generalization of Aumann's result to non-partitional information structures. The two generalizations are used for proving some additional "agreeing to disagree" type results. *Journal of Economic Literature* Classification Number, 026.

1. INTRODUCTION

Aumann, in [1], presented a formalization of the notion of common knowledge and used it to prove that it is impossible "to agree to disagree." That is, given that two players, 1 and 2, agree on the priors, it is impossible that it is common knowledge among the two players that player 1 assigns to some event the probability α_1 and player 2 assigns to the same event the probability α_2 where $\alpha_1 \neq \alpha_2$.

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Aumann's paper was the starting point for a thought provoking literature, which includes also the following two results:

1. Milgrom and Stokey, in [8], proved a result which is often interpreted as referring to the impossibility of speculative trade. Assume that two traders agree on an ex-ante efficient allocation of goods. Then, after the traders get new information, there is no transaction with the property that it is common knowledge that both traders are willing to carry it out.

2. Bacharach in [3] and Samet in [10] explored the validity of Aumann's result where the information structures can be of more general form than in Aumann's model (see also [11]). Recall that in Aumann's model, a player's information structure is described by a partition of the state space. That is, given a particular state of nature, a player's information is captured by the element of the partition which contains that state of nature. Samet showed that Aumann's result continues to hold under weaker assumptions on agents' knowledge that do not necessarily imply that the information structure is described by a partition.

In this note we would like to comment on these developments. We wish to clarify the logic of the "agreeing to disagree" type results, so as to better understand the relations between the work of Aumann, Milgrom and Stokey, Bacharach, and Samet. In particular, we attempt to unify the exposition of these results by tracing them back to properties of functions defined over sets of states of nature (events).

The main observations of the paper are made apparent by considering the following two functions which are defined over sets of states and take the value "True" or "False":

$f_1(X) =$ True iff "the probability of event Z conditional on the event X is α"

$f_2(X) =$ True iff "given the event X, player i prefers action a to action b."

Note that both f_1 and f_2 are such that if they take the value "True" for each of two disjoint sets X and Y, then they take the value "True" for $X \cup Y$. As it turns out, the fact that f_1 has this property provides the main step in proving Aumann's result in a world in which the information structures are described by partitions. Similarly, the main step in the proof of Milgrom and Stokey's result owes to the fact that f_2 has this property.

Note, however, that the functions f_1 and f_2 differ with respect to the following property: if X contains Y and both $f_1(X)$ and $f_1(Y)$ are "True", then $f_1(X - Y)$ is "True," but if $f_2(X)$ and $f_2(Y)$ are "True," then $f_2(X - Y)$ is not necessarily "True." Again, as will be shown, this property of f_1

together with the former property provides the main step in proving the validity of Aumann's result when the information structure is not necessarily partitional. The fact that f_2 does not have this property provides a clue to why Milgrom and Stokey's result fails to hold in a world in which the information structure is not described by a partition.

After presenting the formal concepts in Section 2, we proceed in Section 3 to present a generalization of Aumann's result which has as a special case Milgrom and Stokey's result. Following Samet we prove an analogous result for a more general information structure (Section 4) and then we explain why Milgrom and Stokey's result may not hold for such information structures. The usefulness of the two generalizations is demonstrated by additional examples of "agreeing to disagree" type results.

We would like to emphasize that the purpose of this note is merely to demonstrate the logical structure of the "agreeing to disagree" type results. As such it is only a comment on the existing literature and it is not meant to have any message about the interpretation of the notion of common knowledge (see [9]). The most closely related is the recent work of Geanakoplos [7]. In some sense the points that we make are dual to his results. He was interested in identifying the informational structures for which different "agreeing to disagree" type results hold, while we are interested in identifying the types of theorems which are true only in Aumann's framework and those theeorems which hold in Samet's framework as well.

2. The Model

Let Ω be a finite space of states of nature. Let $i = 1, 2$ be two players. Player i's *information structure* is a function P_i which assigns to all $\omega \in \Omega$ a non-empty subset of Ω. The interpretation of the statement $P_i(\omega) \subset S$ is that at ω agent i knows that the event S occurred. We shall say that i has a *partitional information structure* if there is a partition of Ω such that, for all ω, the set $P_i(\omega)$ is the element in the partition which includes ω.

A set E is called *self evident* if for all $\omega \in E$ and for both i, $P_i(\omega) \subset E$. It is said that the set S is *common knowledge at* ω if there exists a self evident set E such that $\omega \in E$ and $E \subset S$. As was noted in [10], this definition is equivalent to Aumann's definition. Note that, since $\omega \in P_i(\omega)$, a self evident set E satisfies that for both i, $E = \bigcup_{\omega \in E} P_i(\omega)$.

We would like to investigate the possibility of statements of the type: "it is common knowledge that 1 thinks that there is a probability α_1 that event Z has occurred while 2 thinks that the probability is α_2." To formalize such statements, we consider the set F of functions which are defined over the set of subsets of Ω and which assign to every subset of Ω either the value "True" (denoted by T) or the value "False." For $f \in F$ the interpretation of "$f(P_i(\omega)) = T$" is that i knows that the function f gets the value "True" at ω.

We say that f is *preserved under union* if for all sets R and S such that $f(R) = T$ and $f(S) = T$ we have $f(R \cup S) = T$. We say that f is *preserved under disjoint union* if for all disjoint sets R and S such that $f(R) = T$ and $f(S) = T$ we have $f(R \cup S) = T$. We say that f is *preserved under difference* if for all R and S such that $R \supset S$, $f(R) = T$, and $f(S) = T$ we have $f(R - S) = T$.

Following are examples which demonstrate the concepts introduced above. The first two examples generalize Aumann's and Milgrom and Stokey's results. In both examples Ω is endowed with probability measure "pr."

EXAMPLE 1. Let ψ be a random variable defined on Ω and let α be a number. Define the function $f_{\alpha, \psi}$ by $f_{\alpha, \psi}(X) = T$ iff the expectation of ψ given X is α. Observe that $f_{\alpha, \psi}$ is preserved under both disjoint union and difference but not under union. If the expectations of ψ given the disjoint sets X_1 and X_2 are α, hen the expectation of ψ given $X_1 \cup X_2$ is α as well; if the expectations of ψ given sets X_1 and $X_2 \subset X_1$ are α, then the expectation of ψ given $X_1 - X_2$ is also α.

Note that Aumann's setup is a special case of example 1. Let Y be a subset of Ω and denote by $\text{pr}(Y \mid X)$ the probability of Y conditional on X. Consider the random variable $\psi = 1_Y$ (where 1_Y is the characteristic function which gets the value 1 if $\omega \in Y$ and 0 otherwise). Obviously, the expectation of ψ given X is the posterior probability $\text{pr}(Y \mid X)$. Thus, $f_{\alpha, \psi}(X) = T$ iff $\text{pr}(Y \mid X) = \alpha$.

EXAMPLE 2. Define the function $f_{\alpha, \psi}$ by $f_{\alpha, \psi}(X) = T$ iff the expectation of ψ given X is strictly above α (or, alternatively greater or equal to α). Observe that $f_{\alpha, \psi}$ is preserved under disjoint union but not under difference or under union.

Note that Milgrom and Stokey's setup is a special case of Example 2. Let C be a set of consequences. Define a *contingent contract* to be a function from Ω into C. Let A be the set of all contingent contracts. Each of the players has a von Neumann–Morgenstern utility function u_i defined on $C \times \Omega$, i.e., $u_i(c, \omega)$ is player i's utility of the consequence c at state ω. For any $a \in A$ let $U_i(a)$ be the random variable defined by $U_i(a)(\omega) = u_i(a(\omega), \omega)$. Let $a, b \in A$, define ψ_i by $\psi_i = U_i(a) - U_i(b)$, and let $\alpha = 0$. Thus, $f_{\alpha, \psi_1}(X) = T$ iff the expectation of $U_i(a)$ conditional on X is strictly greater than the expectation of $U_i(b)$ conditional on X.

In the next two examples we do not use any probability space on Ω. Instead we refer to an arbitrary function M from Ω into a set N.

EXAMPLE 3. Let $M(\omega)$ be a function from Ω into the set of real numbers and let α be a number. Define the function $f_\alpha(X) = T$ iff α is the median of medians, i.e., the middle of the interval of medians of the vector of numbers $(M(\omega))_{\omega \in X}$. To be more precise, $\alpha = (L + H)/2$ where

$H = \min$ {β: the proportion of numbers in $(M(\omega))_{\omega \in X}$ above β, is below $\frac{1}{2}$}

$L = \max$ {β: the proportion of numbers in $(M(\omega))_{\omega \in X}$ below β, is above $\frac{1}{2}$}.

The function f_α is preserved both under disjoint union and under difference but not under union.

EXAMPLE 4. Let $M(\omega)$ be a function from Ω into the set of real numbers and let n be a number. Define the function $f_n(X) = T$ iff n is the minimum of the function M over the set X. Here, f_n is preserved under union but not under difference.

EXAMPLE 5. Let $M(\omega)$ be a function from Ω into a set N and let $n \in N$. Define the function $f_n(X) = T$ iff there exists an $w \in X$ such that $M(\omega) = n$. Define $g_n(X) = T$ iff $f_n(X) = \text{False}$. Both f_n and g_n are preserved under union but only g_n is preserved under difference.

Finally, let f and g be functions in **F** and define the conjunction function $f \cap g(X) = \text{True}$ iff $f(X) = g(X) = \text{True}$. Note that if f and g are preserved under any of the properties union, disjoint union, or difference, then the function $f \cap g$ is preserved under the respective property.

3. PARTITIONAL INFORMATION STRUCTURES

For the case of partitional information structures, the next proposition unifies a family of "agreeing to disagree" type results, including Aumann's and Milgrom and Stokey's. It is related to a result due to Cave in [5]:

PROPOSITION 1. *Assume that the information structures of the two players are partitional. If f and g are two functions in* **F** *such that*

(1) *for no S both $f(S) = T$ and $g(S) = T$*

(2) *f and g are preserved under disjoint union*

then there is no ω^* for which the set $\{\omega: f(P_1(\omega)) = T$ and $g(P_2(\omega)) = T\}$ is common knowledge at ω^*.

Proof. Suppose that such an ω^* exists. Then, by the definition of common knowledge, there is a self evident set E with $\omega^* \in E$ such that $E \subset \{\omega: f(P_1(\omega)) = T\}$ and $E \subset \{\omega: g(P_2(\omega)) = T\}$. Since the set E is self evident, it is a union of disjoint sets at which f and g get the value "True" and since f and g are preserved under disjoint union, $f(E)=g(E)=T$, a contradiction to (1). ∎

The proposition implies immediately Aumann's "agreeing to disagree" result that there is no ω at which it is common knowledge that player 1 believes that the posterior of X given his information is α_1 and player 2 disagrees with him and believes that the posterior of X given what he knows is $\alpha_2 \neq \alpha_1$. More generally, by applying Proposition 1 to Example 1, we get

CONCLUSION 1. *Let ψ be a random variable on Ω and let α_1 and α_2 be two distinct numbers. There is no ω^* at which it is common knowledge that, conditional on his information, 1 believes that the expectation of ψ is α_1 and, conditional on his information, 2 believes that the expectation is α_2.*

Proof. Recall from Example 1 that $f_{\alpha,\psi}(X)=T$ iff the expectation of ψ given X is α. Note, first, that $f_{\alpha_1,\psi}$ and $f_{\alpha_2,\psi}$ cannot take the value T at the same set and, second, that they are preserved under disjoint union. Therefore, the conclusion follows from Proposition 1. ∎

Now, Aumann's "agreeing to disagree" result is a special case of Conclusion 1. To see this, recall from Example 1 that the posterior probability $\mathrm{pr}(Y|X)$ is the expectation of the random variable $\psi = 1_Y$ conditional on X and hence $f_{\alpha,X}(Y)=T$ iff $\mathrm{pr}(X|Y)=\alpha$.

Proposition 1 also implies Milgrom and Stokey's result. More generally by applying Proposition 1 to Example 2 we get:

CONCLUSION 2. *Let ψ be a random variable on Ω and α a number. Then there is no ω^* at which it is common knowledge that, conditional on his information, 1 believes that the expectation of ψ is strictly above α and, conditional on his information, 2 believes that the expectation of ψ is below α.*

Proof. Define $f_{\alpha,\psi}(X)=T$ iff the expectation of ψ given X is strictly above α and define $g_{\alpha,\psi}(X)=T$ iff the expectation of ψ given X is (weakly) below α. Note, first, that $f_{\alpha,\psi}$ and $g_{\alpha,\psi}$ cannot get the value T at the same set and, second, that they are preserved under disjoint union. Therefore, the conclusion follows from Proposition 1. ∎

Note, however, that it is possible that at ω^* it is common knowledge that player 1 believes that the expectation of ψ is α and player 2 believes that it is different from α. A simple example is $\Omega=\{\omega_1,\omega_2\}$ with equal probabilities, $P_1(\omega)\equiv\Omega$ and $P_2(\omega_1)=\{\omega_1\}$ $P_2(\omega_2)=\{\omega_2\}$, and $\psi(\omega_1)=1$ $\psi(\omega_2)=2$. Then, it is common knowledge that player 1 believes that the expectation of ψ is 1.5 and that player 2 believes that it is 1 or 2.

To address Milgrom and Stokey's result, recall the setup described immediately after example 2. Recall that $U_i(a)$ is the random variable defined by $U_i(a)(\omega) = u_i(a(\omega), \omega)$. Thus, $E[U_i(a) | X]$ is player i's expected utility of the contract a conditional on the set X. Define a contingent contract b to be *ex-ante efficient* if there is no contract a satisfying that, for both i, $E[U_i(a)] > E[U_i(b)]$. Milgrom and Stokey's result is that if b is ex-ante efficient, then there is no ω^* at which the set

$$D^* = \{\omega\colon E[U_i(a) | P_i(\omega)] > E[U_i(b) | P_i(\omega)] \text{ for both } i\}$$

is common knowledge. Assume to the contrary that there is ω^* and a self evident set D such that $\omega^* \in D \subset D^*$. From the definition of self evident set, for all $\omega \in D$, $P_i(\omega) \subset D \subset D^*$. Therefore, for all $\omega \in D$ and for both i, $E[U_i(a) - U_i(b) | P_i(\omega)] > 0$. From the ex-ante efficiency of the contract b, and from $E[U_2(a) - U_2(b) | P_2(\omega)] > 0$, it follows that $E[U_1(a) - U_1(b) | P_2(\omega)] \leqslant 0$. Defining $\psi = U_1(a) - U_1(b)$, we get that for all $\omega \in D$, $E[\psi | P_1(\omega)] > 0$ and $E[\psi | P_2(\omega)] \leqslant 0$. Recall that D is a self evident set and, thus, it is common knowledge at ω^* that $E[\psi | P_1(\omega)] > 0$ and $E[\psi | P_2(\omega)] \leqslant 0$, a contradiction to conclusion 2.

Remark. Milgrom and Stokey's result was extended to other theories of decisions under uncertainty in [3] and [6]. Actually, it is clear from Conclusion 2 that Milgrom and Stokey's result is valid for any theory of choice under uncertainty as long as it satisfies the condition that, if the contract a is preferred to the contract b given two disjoint sets X and Y, then a is also preferred to b given $X \cup Y$.

Proposition 1 provides a scheme for producing more "agreeing to disagree" type results. For example:

CONCLUSION 3. *Let $M(\omega)$ be a function from Ω into the reals and let α_1 and α_2 be two distinct numbers. Then, there is no ω^* such that it is common knowledge at ω^* that player 1 thinks that the median of medians is α_1 and player 2 thinks that the median of medians is α_2.*

CONCLUSION 4. *Let $M(\omega)$ be a function from Ω into the reals and let α_1 and α_2 be two distinct numbers. Then there is no ω^* such that it is common knowledge at ω^* that player 1 thinks that the minimum of M over the set of possible states is α_1 and player 2 thinks that the minimum of M over the set of possible states is α_2.*

Conclusion 4 can be used to analyze Bacharach's "detective" example (see [2]). Assume that two competing detectives are looking for the suspect in a criminal case. From conclusion 4 we get that it is impossible that it is common knowledge that the height of the shortest suspect of detective 1 is n_1, and the height of the shortest suspect of detective 2 is $n_2 \neq n_1$.

Applying proposition 1 to example 5 allows us to conclude a similar "agreeing to disagree" result: it is impossible that it is common knowledge that there is a bespectacled individual among he suspects of detective 1 but not among the suspects of detective 2.

4. Non-partitional Information Structure

Following Bacharach and Samet we turn now to a discussion of the "agreeing to disagree" type results for information structures which are not partitional but satisfy only the following conditions for both i:

(P-1) For all $\omega \in \Omega$, $\omega \in P_i(\omega)$. That is, if i knows the set X then X is true.

(P-2) For all $\omega \in \Omega$ and for all $\omega' \in P_i(\omega)$, $P_i(\omega') \subset P_i(\omega)$. That is, if i knows X he also knows that he knows X.

Let $\mathbf{P}_i = \{S: \exists\omega$ such that $P_i(\omega) = S\}$. As shown by Samet, conditions (P-1, 2) imply that for all R and S in $\mathbf{P}_i$, $R \cap S$ is a union of elements in $\mathbf{P}_i$.

Proposition 2. *Suppose that P_1 and P_2 satisfy conditions* (P-1, 2) *and let f and g be two functions in* $\mathbf{F}$ *such that*

(1) *there is no S for which $f(S) = g(S) = T$*

(2) *f and g are preserved under disjoint union*

and either

(3) *f and g are preserved under difference*

or

(3′) *f and g are preserved under union.*

Then, there is no ω^* at which the set $\{\omega: f(P_1(\omega)) = T$ and $g(P_2(\omega)) = T\}$ is common knowledge.

Proof. Suppose to the contrary that there is a $\omega^* \in \Omega$ and a self evident event E such that $\omega^* \in E \subset \{\omega: f(P_1(\omega)) = T$ and $g(P_2(\omega)) = T\}$. We shall argue that $f(E) = g(E) = T$ and thus get a contradiction. Note that since P_1 satisfies condition (P-1) and the set E is self evident, then $E = E_1 \cup E_2 \cup \cdots \cup E_J$ where for all j, E_j is in $\mathbf{P}_1$. Also, for all E_j, $f(E_j) = T$. If f satisfies the union property it follows immediately that $f(E) = T$. For the case that f satisfies the difference property we will show by induction on the number of elements in D that if $D = D_1 \cup D_2 \cup \cdots \cup D_K$, where, for all k, D_k is in $\mathbf{P}_1$ and $f(D_k) = T$, then $f(D) = T$. Without loss of generality we can assume that for no k and

k' is the set D_k contained in $D_{k'}$. By the inductive hypothesis $f(D_2 \cup \cdots \cup D_K) = T$. If $D_0 = D_1 \cap (D_2 \cup \cdots \cup D_K) = \phi$, then since f is preserved under disjoint union it follows that $f(D) = T$. Otherwise D_0 is a union of elements in $\mathbf{P}_i$ and the number of states in D_0 is strictly smaller than in D. Thus, by the inductive hypothesis, $f(D_0) = T$. Since f is preserved under difference, $f((D_2 \cup \cdots \cup D_K) - D_0) = f(D_1 - D_0) = T$, and since f is preserved under disjoint union, $f(D) = T$. ▮

Note that Proposition 2 can be strengthened in two senses. First, we can replace the requirement that either (3) or (3′) hold with a weaker condition that for all Q and R such that $f(Q) = T$ and $f(R) = T$, either $[f(Q \cup R) = T]$ or $[f(Q \cap R) = T$ implies $f(Q - R) = T$ or $f(R\text{-}Q) = T]$. Second, it is actually straightforward that for functions which are preserved under union, Proposition 2 holds even for information structures which satisfy only condition (P-1).

It follows from Proposition 2 that, since the functions in Conclusions 1 and 3 are preserved under difference as well, these conclusions continue to hold for information structures that satisfy conditions (P-1, 2) even if they are not partitional. In particular, as Samet showed, Aumann's result holds in this case as well. (Note that the proof of Proposition 2 used the finiteness of Ω, while Samet extended Aumann's result for the more complicated setting of infinite state space.) Conclusion 4 holds since the functions used in its statement are preserved under union.

The functions considered in Conclusion 2 are not preserved under difference or under union. Indeed, the following example demonstrates that Conclusion 2 is not necessarily true if the information structure is not partitional. To see this consider the space $\Omega = \{\omega_1, \omega_2, \omega_3\}$ where all states are equally likely. Assume

$$P_1(\omega) \equiv \{\omega_1, \omega_2, \omega_3\}$$

$$P_2(\omega_1) = \{\omega_1, \omega_2\}, \qquad P_2(\omega_2) = \{\omega_2\}, \qquad P_2(\omega_3) = \{\omega_2, \omega_3\}.$$

Both P_i satisfy conditions (P-1, 2) and P_2 is not partitional.

A Counter Example to Conclusion 2. Let ψ be the random variable $\psi(\omega_2) = 1$ and $\psi(\omega_1) = \psi(\omega_3) = 0$ and pick $\alpha = 0.35$. For all ω it is common knowledge that 1 believes that the expectation of ψ is $\frac{1}{3}$ (which is less than 0.35) and that 2 believes that the expectation of ψ is 0.5 or 1 (which are strictly above 0.35).

A more direct counter example for Milgrom and Stokey's result is as follows.

Let $B = \{a, b\}$ and let u_1 and u_2 be von Neumann–Morgenstern utilities presented by the following table:

	Player 1		Player 2	
	a	b	a	b
ω_1	3	0	0	3
ω_2	0	5	5	0
ω_3	3	0	0	3

The contingent contract $x(\omega) \equiv b$ is ex ante efficient, but for all ω it is common knowledge that $y(\omega) \equiv a$ is preferred by both players to $x(\omega)$.

The observation that Conclusion 2 does not hold for the information structure without partitions appears first in [4].

5. Conclusion

"Agreeing to disagree" type results can be traced back to simple properties of functions defined over subsets of the state space: being preserved under union, disjoint union, and difference. In this paper we exposed some of the logic of those results by showing that different properties of informational structures are closely linked with different properties of the functions used in the statements of the propositions. As a byproduct of this analysis we showed that these observations can be used not only to unify the proofs of existing results, but also to generate new results of this type.

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