

# The Nash bargaining solution in economic modelling

Ken Binmore\*

Ariel Rubinstein\*\*

and

Asher Wolinsky\*\*

*This article establishes the relationship between the static axiomatic theory of bargaining and the sequential strategic approach to bargaining. We consider two strategic models of alternating offers. The models differ in the source of the incentive of the bargaining parties to reach agreement: the bargainers' time preference and the risk of breakdown of negotiation. Each of the models has a unique perfect equilibrium. When the motivation to reach agreement is made negligible, in each model the unique perfect equilibrium outcome approaches the Nash bargaining solution with utilities that reflect the incentive to settle and with the proper disagreement point chosen. The results provide a guide for the application of the Nash bargaining solution in economic modelling.*

## 1. Introduction

■ This article studies the relations between the static axiomatic and the dynamic strategic approaches to bargaining. Its purpose is to clarify certain interpretive ambiguities in the axiomatic approach to provide a more solid grounding for applications of the Nash bargaining solution in economic modelling. The article may therefore be seen as a contribution to the "Nash program" as described by Binmore (1980, 1985).

In a two-person bargaining situation, there is a set  $X$  of possible agreements, where  $x \in X$  specifies the physical consequences to the two parties if  $x$  is agreed upon by both. Other information that might be relevant to the problem includes the parties' preferences over  $X$ , their attitudes toward risk and time, the bargaining procedure (e.g., who offers to whom and when), and the environment within which the bargaining takes place (e.g., can the process be interrupted by random events?).

The static axiomatic approach, which has its origins in the work of Nash (1953), describes the bargaining problem by using *only* the information contained in a pair of utility functions  $u_1, u_2$ , which represent the parties' preferences over  $X$ , and a pair of utility levels that are referred to variously as the *status quo*, the disagreement point, or the threat point. That is,

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\* London School of Economics.

\*\* The Hebrew University, Jerusalem.

the bargaining problem is represented by the pair  $(S, s^0)$ , where  $S = \{(u_1(x), u_2(x)): x \in X\}$  and  $s^0 \in S$ .

The choices of  $u_1$ ,  $u_2$ , and  $s^0$  are matters of modelling judgment. As we know from the work of Shapley (Roth, 1979), one cannot base a meaningful two-person bargaining solution on the parties' ordinal preferences over  $X$  alone. Therefore, the first modelling judgment concerns the additional information (e.g., time preferences or attitudes toward risk) that is to be embedded in the functions  $u_i$ . The second modelling judgment concerns the choice of  $s^0$ . There could be a number of elements in the underlying situation that are natural candidates for this role (e.g., the best outside alternatives for the parties if they withdraw from the bargaining or the possibility that they keep bargaining forever).

Within the static axiomatic approach, the best-known description of the bargaining situation is Nash's bargaining model. Here the functions  $u_i$  are the von Neumann-Morgenstern utility representations of the parties' preferences. That is, the extra information embedded in the functions  $u_i$  concerns the parties' attitudes toward risk. But the precise meaning of  $s^0$  in Nash's model is somewhat vague. The solution to Nash's problem is the unique element in  $S$  that satisfies a number of axioms. It turns out that this element is the argument that maximizes the Nash product  $(s_1 - s_1^0)(s_2 - s_2^0)$ .

But the parties' attitudes toward risk are not the only piece of additional information that can be used for the choice of  $u_i$  and hence the construction of  $S$ . For example, in what follows we shall consider a problem  $(S, s^0)$ , where the functions  $u_i$  used in its construction are chosen on the basis of information about the time preferences of the parties.

Most information concerning the bargaining procedure and the environment within which bargaining operates is abstracted away in the static axiomatic approach.<sup>1</sup> The dynamic strategic approach attempts to treat concretely these missing elements. It is based on the construction of noncooperative bargaining games that describe the bargaining process explicitly (Rubinstein, 1982). A unique perfect equilibrium outcome of such a game is then viewed as the solution to the bargaining situation studied.

The main purpose of our article is to use the insights of the strategic approach in selecting "appropriate" static representations  $(S, s^0)$  for certain common bargaining situations. We hope that this will aid the economic modelling of bargaining problems for which the Nash bargaining solution is currently used rather mechanically, without the choice of the static representation  $(S, s^0)$  tailored to the underlying bargaining situation.

As an example of the problems that may arise in this type of modelling, consider the familiar application of the Nash bargaining solution to wage negotiations over income streams (Ellis and Fender, 1985; Grout, 1984; McDonald and Solow, 1981). First, as indicated above, the set  $S$  in Nash's model is constructed on the basis of information concerning the parties' attitudes toward risk. This could be an appropriate choice if random events play an important role in the underlying bargaining process. It could be, however, that the bargaining takes place in an essentially deterministic environment and that what motivates the parties to reach an agreement are the losses associated with delays (e.g., the income foregone by the employee and the employer during the dispute). Then the parties' attitudes toward risk are not immediately relevant, and it seems appropriate to construct the set  $S$  on the basis of information about the parties' time preferences. Next, consider the ambiguities that may arise in locating  $s^0$ . One possibility is to identify  $s^0$  with the streams of income accruing to the two parties in the course of the dispute. For example, if the dispute involves a strike, these income streams would be the employee's income from temporary work, union strike funds, and similar sources, while the employer's income might derive from temporary arrangements that keep the business running. Another possibility is to identify  $s^0$  with the income streams available to the parties if they abandon the attempt to reach an agreement

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<sup>1</sup> One exception perhaps is the symmetry axiom, which can also be interpreted as a referring to a symmetry in the procedure.

and take up the best permanent alternative elsewhere. In the case of the employee, this might be his income stream in an alternative job. For the employer, it might be the income stream derived from using a less skilled worker.

The example demonstrates the need for a theory that will help to resolve this modelling problem. We attempt to tackle the problem in two stages. First, we present noncooperative bargaining models that describe the bargaining process explicitly and capture what in our opinion are important features of bargaining situations. We can think of two basic motives that may induce parties to a bargaining process to reach an agreement rather than to insist indefinitely on incompatible demands. One motive has to do with the parties' impatience to enjoy the fruits of an agreement. The other motive has to do with the parties' fear that, if they prolong the negotiations, they might lose the opportunity to reach an agreement at all. Accordingly, we adopt strategic bargaining models that capture, in turn, each of these motives. The first motive is captured by a natural model considered by Rubinstein (1982). In this model the bargaining takes place over time according to a predetermined procedure of alternating offers and responses of both parties. The parties' incentive to agree lies in the fact that they are impatient. The second motive is captured by another version of the alternating offers model. In this version the parties are not impatient, but they face a risk that if agreement is delayed, then the opportunity they hope to exploit jointly may be lost (e.g., while they are bargaining the opportunity might be snatched by a third party). The second version is closer to Nash's (1953) own attempt to justify his axiomatic solution with a noncooperative game (see also Crawford (1982) and Moulin (1984)). It differs from Nash's attempt in not postulating an unrealistic (Schelling, 1960) capacity on the part of the players to make binding threats.

The two strategic models exhibit unique perfect equilibria. Following Binmore (1980), we allow the time interval between successive offers in both models to decrease to zero. This allows a study of the limiting situations in which the bargaining procedure is essentially symmetric or the potential costs of delaying agreement by one period can be regarded as negligible.

Second, we construct two static problems  $(S, s^0)$ , which correspond to the two bargaining situations. That is, in each case we suggest a choice of the utility functions  $u_1$  and  $u_2$  used in defining  $S = \{(u_1(x), u_2(x)): x \in X\}$  and of an element  $s^0 \in S$ . In the second model, where the driving force is the probability of breakdown,  $u_1$  and  $u_2$  are derived from the parties' attitudes toward risk, i.e., they are the von Neumann-Morgenstern utility functions;  $s^0$  corresponds to the outcome obtained in the event of a breakdown of the bargaining process. In the first model, where the driving force is the parties' impatience, we use a representation result of Fishburn and Rubinstein (1982) to derive the functions  $u_1$  and  $u_2$  from the parties' time preferences. These utility representations reflect the relative impatience of the parties. The element  $s^0$  corresponds to the outcome that has the property that each of the parties is indifferent between reaching this outcome now or reaching it at any future time. That is,  $s^0$  has the interpretation of the *status quo* (no loss-no gain) positions of the parties.

Having constructed the pairs  $(S, s^0)$ , we show that in each of the models the limiting equilibrium outcome coincides with the solution of the respective maximization problem:  $\max (s_1 - s_1^0)(s_2 - s_2^0)$ . Thus, the second model implements *the standard Nash solution*, given  $s^0$ , while the first model implements what we shall call the *time-preference Nash solution*. Notice that although the two solutions are the unique points at which the Nash product is maximized for the relevant pair  $(S, s^0)$ , we refer to them by distinct names to emphasize that their implementation will lead to *different* agreements with respect to the underlying set  $X$  of *physical* consequences to the two parties.

We should note that Binmore (1980) has already observed the link between the limiting equilibrium outcome in Rubinstein's model and the Nash solution. The present work, however, shows that this limiting outcome is actually not the Nash solution as normally interpreted, but rather is the inherently different time-preference Nash solution.

We believe that in certain situations these results provide some firm theoretical foundations for the economic modelling practice of treating bargaining problems in the static form  $(S, s^0)$  and applying the axiomatic Nash bargaining solution.

Furthermore, our results offer some clearer insights into some results and practices that have developed in the static axiomatic bargaining literature. One such insight concerns the familiar exercise of applying a concave transformation to the utility function of one of the parties in the static representation and observing that this changes the solution in favor of the other party. It follows from our analysis that this observation carries different interpretations in the two models. In both models the concave transformation weakens the bargaining position of the adversely affected party—in the impatience model because the party was made more impatient, while in the risk model because the party was made more risk averse. The traditional interpretation has referred only to the change in attitudes toward risk (Kihlstrom, Roth, and Schmeidler, 1981), even when the bargaining process was driven by the parties' impatience (Roth, 1985), although there is no apparent reason that a party's attitudes toward risk should affect his bargaining position in a riskless environment.

Another insight gained by our analysis concerns the application of the asymmetric Nash solutions for which  $(s_1 - s_1^0)^\beta (s_2 - s_2^0)^{1-\beta}$  is maximized (Roth, 1979; Binmore, 1980). Our method points to these solutions when there is asymmetry in the bargaining procedure or in the parties' beliefs. These include, for example, the case of longer waiting time after one party's offer than after the other's offer and the case of different beliefs concerning the probability of breakdown in the risk model.

The rest of the article is organized as follows. Section 2 reviews the strategic bargaining models we use in the analysis. After introducing the common basic model, we describe the strategic model in which the incentive to reach agreement is due to the impatience of the parties and then the model in which the incentive to reach an agreement is provided by the risk of breakdown. In Section 3 we show that the Nash bargaining solution is an approximation to the perfect equilibria in the two strategic models of Section 2. Section 4 guides the modeller in fitting a Nash bargaining problem to an underlying situation. We discuss the choice of utilities, then identify the disagreement point, and assess the results when there are asymmetries in the bargaining procedure. Concluding remarks appear in Section 5.

## 2. Strategic bargaining models

■ This section reviews the strategic bargaining models we shall use in the subsequent analysis. All the models are based on the strategic alternating offers model presented by Rubinstein (1982), and, therefore, we omit the proofs and refer the interested reader to that article.

□ **The basic model of strategic bargaining.** Two players, denoted by 1 and 2, have the opportunity to reach an agreement. The set of possible agreements is

$$X = \{(x_1, x_2) | x_1, x_2 \geq 0, x_1 + x_2 \leq 1\}.$$

There is also an outcome  $d$ , which represents the possibility that the players never reach an agreement.

The bargaining takes place over time and the players follow a predetermined bargaining procedure. The moves are made at points of time,  $0, \Delta, 2\Delta, 3\Delta, \dots$ , where  $\Delta$  is the length of a single bargaining period. At each of these points one of the players suggests an agreement  $x \in X$ , and his opponent can either accept it or reject it. Acceptance of agreement  $x$  concludes the bargaining with the agreement  $x$ , while upon rejection the process continues to the next period. We assume that player 1 starts the process and that subsequent offers are made in alternating order.

A strategy  $f$  for a player is a sequence of rules,  $f = (f^t)_{t=0}^\infty$ , where each rule  $f^t$  describes the player's move at time  $t\Delta$ . A player's move is either an offer or a reaction, depending on

whose turn it is to make an offer at that time. Each  $f^t$  may depend on the entire history of the process up to time  $t\Delta$ . A typical outcome of the game is either  $(x, t\Delta)$ , which means that the agreement  $x$  is reached at time  $t\Delta$ , or the outcome  $d$ .

To complete the description of the game, we must specify the players' preferences over agreements and their timing. First, we describe the players' basic preferences over  $X \cup \{d\}$  (i.e., their preferences over agreements that are reached immediately or their preferences if they are indifferent to the timing). In the subsequent models we shall extend this description to capture the players' preferences over the timing of agreements as well. Let  $\succeq_i$  denote player  $i$ 's preference ordering over  $X \cup \{d\}$ . We make the following assumptions.

*Assumption 1.* There is a conflict of interests:  $(x_1, x_2) \succeq_i (y_1, y_2)$  if and only if  $x_i \geq y_i$ .

*Assumption 2.* There are mutually beneficial agreements:  $x \in X$  such that  $x >_i d$ ,  $i = 1, 2$ .

If the players are indifferent to the timing of the agreement, the above describes a game in extensive form. It is easy to see the following proposition.

*Proposition 1.* If the players are indifferent to the timing of the agreement, then the set of perfect equilibrium outcomes for this game includes every  $x \in X$  such that  $x \succeq_i d$ ,  $i = 1, 2$ .

Thus, when the bargaining process is "frictionless" in the sense that delays are costless, every individually rational outcome is a perfect equilibrium outcome. In what follows we introduce different types of imperfections into the process by explicitly modelling the losses associated with delayed agreements.

□ **A strategic bargaining model with time preferences.** In our first model the extensive form of the bargaining game is the same as in our basic model, but here the players are induced to reach an agreement by their impatience for the outcomes. The players' impatience is captured by extending the preference orderings  $\succeq_i$  to preference orderings over  $(X \times T) \cup \{d\}$ , where  $T = [0, \infty)$  is the time space. As before, the pair  $(x, \tau)$  will denote "agreement  $x \in X$  is reached at time  $\tau \in T$ ." The preferences over the outcomes of the form  $(x, 0)$  are required to satisfy Assumptions 1 and 2. In addition we assume that Assumptions 3–7 hold.

*Assumption 3.* There are time-indifferent agreements: there exists a  $g \in X$  such that  $\forall \tau \in T, (g, \tau) \sim_i (g, 0)$ .

Assumption 3 requires that the set  $X$  is rich enough that for each player there exists a *status quo* agreement (no loss-no gain)  $g$ . In principle, player  $i$  need not be indifferent between  $(g, 0)$  and  $d$ , but it seems that in most interesting examples  $(g, 0) \sim_i d$ . Note that this assumption is violated if the parties have per unit time bargaining costs.

*Assumption 4.* There is stationarity: for every  $x, y \in X, \tau, \tau' \in T, v \geq 0$ , if

$$(x, \tau) \succeq_i (y, \tau + v),$$

then  $(x, \tau') \succeq_i (y, \tau' + v)$ .

*Assumption 5.* There is monotonicity in time: for every  $x \in X, \tau_1 < \tau_2 \in T$ , if  $(x, 0) >_i (g, 0)$ , then  $(x, \tau_1) >_i (x, \tau_2)$ .

*Assumption 6.* There is continuity: the graph of  $\succeq_i$  is closed.

*Assumption 7.* Compensation is concave. Let the function  $c_1$  be defined by  $(x + c_1(x), \Delta) \sim_1 (x, 0)$ ; then  $c_1$  is increasing and concave. (The analogous assumption is made for  $\succeq_2$  as well.)

The next proposition characterizes the unique perfect equilibrium of this game. It is a direct application of the main theorem in Rubinstein (1982), and therefore we omit the proof.

*Proposition 2.* (a) There exists a single pair  $x^*, y^* \in X$  such that  $(x^*, \Delta) \sim_1 (y^*, 0)$  and  $(y^*, \Delta) \sim_2 (x^*, 0)$ . (b) If both  $x^*$  and  $y^*$  are preferred by both players to  $g$  and  $d$ , then the game with the time preferences has a unique perfect equilibrium. The equilibrium strategies are such that player 1 always demands  $x^*$  and rejects any offer strictly below  $y^*$ , and player 2 always offers  $y^*$  and rejects any offer strictly above  $x^*$ .

Since player 1 begins, the equilibrium outcome is  $(x^*, 0)$ . If player 2 began, it would be  $(y^*, 0)$ .

□ **A strategic model with exogenous risk of breakdown.** In this second model we introduce a different type of imperfection into the bargaining process. Here the time itself is not valuable, but in each passing period there is an exogenous risk that the bargaining process will terminate without an agreement. For example, the players bargain over the gains from some opportunity that they cannot exploit without reaching an agreement, but in the meantime this opportunity might be exploited by a third party. It is assumed that, conditional on the bargaining process' reaching time  $\tau$  and no agreement's being reached before time  $\tau + h$ , the probability that the process will break down before time  $\tau + h$  is  $\lambda h + o(h)$ . That is, the time of the breakdown is exponentially distributed with parameter  $\lambda$ . Thus, in each bargaining period of length  $\Delta$  that separates two consecutive bargaining stages there is a positive probability  $p = p(\Delta) = 1 - e^{-\lambda\Delta}$  that the process will break down, in which case the outcome will be  $b \in X$ . For the purpose of this model, the preferences of our basic model have to be extended to preference orderings,  $\succeq_i$ , on the set of lotteries over elements of  $X$ . We assume that in addition to Assumptions 1 and 2, the orderings  $\succeq_i$  satisfy Assumptions 8 and 9.

*Assumption 8.* The orderings  $\succeq_i$  can be represented by the expected values of continuous utility functions  $u_i: X \rightarrow R$ . These are the von Neumann-Morgenstern assumptions.

*Assumption 9.* The preferences  $\succeq_i$  display risk aversion: for every  $x, y \in X$ ,  $\alpha \in [0, 1]$ ,  $\alpha x + (1 - \alpha)y \succeq_i \alpha x \oplus (1 - \alpha)y$ , where  $\alpha x \oplus (1 - \alpha)y$  denotes the lottery whose outcomes are  $x$  and  $y$  with probabilities  $\alpha$  and  $1 - \alpha$ , respectively.

Suppose now that the players employ strategies such that if the bargaining process does not break down before time  $t\Delta$ , it will be concluded at  $t\Delta$  with agreement  $x$ . We denote the outcome induced by this strategy pair as  $\langle x, t\Delta \rangle$ . Thus,  $\langle x, t\Delta \rangle$  is the lottery

$$\langle x, t\Delta \rangle = (1 - p)^t x \oplus (1 - (1 - p)^t) b.$$

The next proposition asserts the existence of the unique perfect equilibrium of this game and characterizes that equilibrium.

*Proposition 3.* (a) There exists a unique pair  $x^*, y^* \in X$  such that  $\langle x^*, \Delta \rangle \sim_1 \langle y^*, 0 \rangle$  and  $\langle y^*, \Delta \rangle \sim_2 \langle x^*, 0 \rangle$ . (b) There exists a unique perfect equilibrium. The equilibrium strategies are those described in Proposition 2.

*Proof.* By Assumption 8 the preference ordering  $\succeq_i$  over lotteries  $\langle x, t\Delta \rangle$  can be represented by the function

$$(1 - p)^t u_i(x) + (1 - (1 - p)^t) u_i(b).$$

These functions induce preference orderings over  $X \times T$  that satisfy Assumptions 3–7, and therefore Proposition 3 is a special case of Proposition 2. *Q.E.D.*

### 3. Nash solution as an approximation to the equilibria of the strategic models

■ The dynamic strategic models reviewed above provide detailed descriptions of bargaining processes that may lie in the background of static axiomatic models. This section studies the relations between the noncooperative solutions of the dynamic strategic models and the Nash bargaining solution of corresponding static axiomatic models. First, we examine the limiting perfect equilibrium outcomes, which are obtained when the length  $\Delta$  of a single bargaining period approaches zero. Then we fit a pair  $(S, s^0)$  to each strategic model. This requires making a meaningful choice of the utility representations  $u_i: X \rightarrow R, i = 1, 2$ , which define  $S = \{(u_1, u_2) | (u_1, u_2) = (u_1(x), u_2(x)), x \in X\}$ , and then selecting an appropriate element  $s^0 \in S$ . Finally, we show that the agreement  $x_N = \arg \max_x (u_1(x) - s_1^0)(u_2(x) - s_2^0)$ , which is specified by the Nash solution, coincides with the limiting perfect equilibrium outcome.

We should emphasize that the choice of  $u_i$  and  $s^0$  is not made separately for each special case just to achieve the correspondence between the limiting perfect equilibrium and the Nash solution. Rather, for all strategic models of a particular type (the time-preference models and the exogenous-risk models), the choice of  $u_i$  and  $s^0$  is made by using the same method. Furthermore, in each case these choices have a natural interpretation: for the time-preference model the choice of the  $u_i$  reflects the relative degrees of impatience of the two parties; for the exogenous-risk model the  $u_i$  are the von Neumann-Morgenstern representations, and thus reflect the parties' relative degrees of risk aversion.

□ **Time-preference Nash solution as an approximation to the equilibrium outcome of the strategic model with time preferences.** This subsection points out the exact relationship between the perfect equilibrium of the time-preference model and the Nash solution. Recall that in that model the preferences  $\succeq_i$  of the parties were defined over the set  $(X \times T) \cup d$ . Time preferences of the type considered here were studied by Fishburn and Rubinstein (1982), who have established the following results. First, if the ordering  $\succeq_i$  satisfies Assumptions 1–6, then for any  $\delta \in (0, 1)$  there exists a continuous function  $u_i(x)$  such that  $\succeq_i$  is represented by the utility function  $\delta^t u_i(x)$ :  $(x, t) \succeq_i (y, s)$  if and only if  $\delta^t u_i(x) \geq \delta^s u_i(y)$ . Second, given  $\delta$ , the function  $u_i(x)$  is unique up to multiplication by a positive constant.<sup>2</sup> Third, if both  $u_i(x)\delta^t$  and  $v_i(x)\epsilon^t$  represent the ordering  $\succeq_i$ , then there exists a constant  $K_i > 0$  such that

$$v_i(x) = K_i [u_i(x)]^{\log \epsilon / \log \delta}. \quad (1)$$

Fourth, if  $\succeq_i$  also satisfies Assumption 7, then there exists  $\hat{\delta}_i$  such that all functions  $u_i(x)$  that correspond to a value of  $\delta \geq \hat{\delta}_i$  are concave.

We shall now use the above-quoted results to construct a pair  $(S, s^0)$  from the data of the strategic model considered here. Given  $\succeq_i$ , choose a discount factor  $\delta \geq \max(\hat{\delta}_1, \hat{\delta}_2)$ , and let  $u_i(x)\delta^t$  be the utility representation of  $\succeq_i, i = 1, 2$ . Define  $S = \{(u_1, u_2) | x \in X \text{ subject to } (u_1, u_2) = (u_1(x), u_2(x))\}$  and  $s^0 = (0, 0)$ . Let the function  $u_2 = \psi(u_1)$  describe the frontier of the set  $S$ . Since the functions  $u_i(x)$  are strictly increasing, continuous, and concave, the function  $\psi$  is continuous, strictly decreasing, and concave on  $[u_1(0), u_1(1)]$ . Given  $(S, s^0)$ , let  $x_N \in X$  be defined by

$$x_N = \arg \max_{x \in X} u_1(x)u_2(x).$$

<sup>2</sup> The utility representation is in fact unique only up to separate multiplicative rescalings of the positive and negative utilities. Since only the positive parts of the utilities matter for the present analysis, however, this is of no consequence.

Notice that  $x_N$  is independent of the particular choice of  $\delta$  and the corresponding  $u_i$ ,  $i = 1, 2$ . The reason is that, given any other common discount factor  $\epsilon \in (0, 1)$  and the corresponding utilities  $\epsilon^i v_i(x)$ , we have from (1) that

$$v_1(x)v_2(x) = K_1K_2[u_1(x)u_2(x)]^{\log\epsilon/\log\delta},$$

and, therefore,

$$\arg \max_x v_1(x)v_2(x) = \arg \max_x u_1(x)u_2(x) = x_N.$$

To emphasize the fact that  $x_N$  depends on the parties' time preferences, we shall denote the time-preference Nash solution as  $x_N^{TP}(\succsim_1, \succsim_2)$ .

The following proposition asserts that  $x_N^{TP}(\succsim_1, \succsim_2)$  approximates the perfect equilibrium outcome of our time-preference model when the length of a single bargaining period,  $\Delta$ , is sufficiently small. This result is an extension and modification of a result due to Binmore (1980). We prove it in the Appendix.

*Proposition 4.* Let  $x^*(\Delta)$ ,  $y^*(\Delta)$  be the unique pair of agreements defined in Proposition 2, so that  $x^*(\Delta)$  is the perfect equilibrium outcome. Then

$$\lim_{\Delta \rightarrow 0} x^*(\Delta) = \lim_{\Delta \rightarrow 0} y^*(\Delta) = x_N^{TP}(\succsim_1, \succsim_2).$$

Note that, in the limit, it is irrelevant who makes the opening demand.

□ **Nash solution as an approximation to the equilibrium outcome of the strategic model with uncertain termination of bargaining.** This subsection points out the exact relationship between the perfect equilibrium of the exogenous-risk model and the Nash solution. Recall that in the exogenous-risk model the preferences of the parties,  $\succsim_i$ , were defined on the set of all lotteries over  $X$ . To construct a Nash bargaining problem,  $(S, s^0)$ , on the basis of the data of this model, let  $u_1(x)$  and  $u_2(x)$  be the von Neumann-Morgenstern utility representations of the preferences  $\succsim_1$  and  $\succsim_2$ , and define

$$S = \{(u_1, u_2) | x \in X \text{ subject to } (u_1, u_2) = (u_1(x), u_2(x))\}$$

$$s^0 = (u_1(b), u_2(b)).$$

Notice that Assumption 9 implies that both  $u_1$  and  $u_2$  are concave, so that the frontier of  $S$ ,  $u_2 = \psi(u_1)$ , is concave as well.

The Nash solution  $(u_1^N, u_2^N)$  of the problem  $(S, s^0)$  described above is the solution for

$$\max_{(u_1, u_2) \in S} (u_1 - u_1(b))(u_2 - u_2(b)).$$

Therefore, the agreement  $x_N$  specified by the Nash solution is

$$x_N = \arg \max_{x \in X} [u_1(x) - u_1(b)][u_2(x) - u_2(b)].$$

It is well known and easily verified that  $x_N$  is independent of the particular von Neumann-Morgenstern utility representation chosen; that is,  $x_N$  is not affected by affine transformations of the  $u_i$ 's. To emphasize this fact we shall write the von Neumann-Morgenstern Nash solution as  $x_N^{NM}(\succsim_1, \succsim_2)$ .

The following proposition asserts that when the length  $\Delta$  of a single bargaining period is sufficiently small—the probability  $p(\Delta)$  of a breakdown between consecutive bargaining sessions is small—the Nash solution  $x_N^{NM}(\succsim_1, \succsim_2)$  approximates the perfect equilibrium outcome of the exogenous-risk model.

*Proposition 5.* Let  $x^*(\Delta)$  and  $y^*(\Delta)$  be the unique pair of agreements defined in Proposition 3, so that  $x^*(\Delta)$  is the perfect equilibrium outcome. Then

$$\lim_{\Delta \rightarrow 0} x^*(\Delta) = \lim_{\Delta \rightarrow 0} y^*(\Delta) = x_N^{NM}(\succsim_1, \succsim_2).$$

*Proof.* The result follows from the fact that this proposition is formally a special case of Proposition 4. To see this, recall that the preferences over the lotteries  $\langle x, t\Delta \rangle$  are represented by

$$(1 - p(\Delta))^t u_i(x) + [1 - (1 - p(\Delta))^t] u_i(b), \quad i = 1, 2,$$

where  $p(\Delta)$  is the probability of a breakdown in the course of a single bargaining period. Therefore, the preferences are also represented by

$$(1 - p(\Delta))^t [u_i(x) - u_i(b)], \quad i = 1, 2. \quad (2)$$

Recall that  $p(\Delta) = 1 - e^{-\lambda\Delta}$ , and define  $\delta = e^{-\lambda}$  so that (2) can be written as

$$\delta^{t\Delta} [u_i(x) - u_i(b)], \quad i = 1, 2. \quad (3)$$

These functions can be viewed as if they were induced by preference orderings over  $X \times T$  that satisfy Assumptions 3–7. Now, since  $x^*(\Delta)$  and  $y^*(\Delta)$  characterize the perfect equilibrium of the strategic time-preference model with the time preferences that induce (3), we have from Proposition 4 that

$$\lim_{\Delta \rightarrow 0} x^*(\Delta) = \lim_{\Delta \rightarrow 0} y^*(\Delta) = \arg \max_x [u_1(x) - u_1(b)][u_2(x) - u_2(b)] = x^{NM}(\bar{z}_1, \bar{z}_2). \quad Q.E.D.$$

#### 4. A guide to applications

■ In the previous sections we presented two types of bargaining situations, the time-preference model and the exogenous-risk model, which capture in turn the two major motives that induce bargaining parties to reach an agreement. In each case we showed how the data of the underlying situation can be used to construct a pair  $(S, s^0)$  such that the solution for  $\max_{s \in S} (s_1 - s_1^0)(s_2 - s_2^0)$  approximates the unique perfect equilibrium outcome of a natural

strategic model. The contribution of our analysis to the modelling of bargaining situations is twofold. First, it lends further support to the use of the relatively simple Nash solution in economic applications. Second, it guides the modeller in the subtle task of fitting the representation  $(S, s^0)$  to the underlying situation. In what follows we summarize the main insights that are relevant for modelling.

□ **Construction of the set  $S$ .** The construction of the set  $S$  amounts to choosing utility functions  $u_1$  and  $u_2$  such that  $S = \{(u_1(x), u_2(x)) | x \in X\}$ . Our method suggests that the choice of  $u_1$  and  $u_2$  should reflect the central motive that drives the bargaining parties to reach an agreement in the modelled situation. If the modelled bargaining process takes place in a riskless environment and the parties are motivated by their impatience for the outcome, then the functions  $u_1$  and  $u_2$  should be derived from the time preferences of the parties as in Section 3's discussion of the time-preference model. If, however, the central motive to reach an agreement is provided by the risk of breakdown in the negotiations, then the functions  $u_1$  and  $u_2$  should be derived from the parties' attitudes towards risk, the von Neumann-Morgenstern utility representations.

The meaning of the selected utility functions determines the appropriate interpretation of comparative-statics results obtained in the static model. Consider, for example, the familiar exercise of applying a concave transformation to the utility function of one of the parties (say,  $u_2$  is replaced by  $v_2(x) = h[u_2(x)]$ , where  $h$  is concave) and observing that this changes the Nash solution in favor of the other party (Kihlstrom, Roth, and Schmeidler, 1981). In both models the interpretation is that the concave transformation weakens the bargaining position of the adversely affected party. In the impatience model, where the utility functions are derived from the time preferences, this is so since the adversely affected party was made more impatient (the function  $v_2$  represents a preference ordering over  $X \times T$  that displays greater impatience than the ordering represented by  $u_2$ ). In the risk model, where the utility

functions are the von Neumann-Morgenstern utility representations, this is so since the party was made more risk averse.

In contrast, the common interpretation given in the literature is to attribute this comparative-statics result to a change in the party's attitudes toward risk, even when the driving force is impatience (Roth, 1985). This is done despite the fact that in the context of the impatience model, there is no reason why attitudes toward risk should affect the strength of a bargaining position.

□ **The choice of  $s^0$ .** The Introduction described some of the ambiguities that may arise in locating the element  $s^0$ . The strategic time-preference and exogenous-risk models in Section 2 provide a useful guide for the identification and interpretation of  $s^0$  in static models. For the time-preference model, the analysis in Sections 2 and 3 points to the choice of  $s^0 = (0, 0) = (u_1(g), u_2(g))$ . The agreement  $g$  has the interpretation of *status quo* agreement: no loss-no gain as compared with the parties' positions in the course of the negotiations. In the example of the wage negotiations considered in the Introduction, the outcome  $g$  is identified with the agreement that gives the parties the same income streams as they are receiving during the dispute. For the model of the risk of breakdown of bargaining in Sections 2 and 3, the analysis points to the choice of  $s^0 = (u_1(b), u_2(b))$ , where  $b$  is the outcome in the event that the bargaining process does break down.

The same type of reasoning that led us to identify  $s^0$  with  $g$  or  $b$  can be used to reject the common interpretation of  $s^0$  as the outside options of the bargaining parties in most cases of practical interest. An outside option is defined to be the best alternative that a player can command if he withdraws unilaterally from the bargaining process. Usually it is assumed that there exists an outcome  $e \in X$  (the "outside option point") that results if either bargaining party withdraws from the bargaining process, although a more general description might specify two outcomes, depending on who withdraws. It is reasonable to identify  $s^0$  with  $e$  if the bargaining procedure makes "take it or leave it" threats credible as in Nash's (1953) demand game (see also Crawford (1982)). But, as Schelling (1960) has emphasized, this assumption is seldom realistic, and, indeed, it is to avoid such assumptions that an analysis in terms of perfect equilibria is necessary.

Outside options may be incorporated with the strategic models of Section 2 by modifying the extensive form of the game so that at each node of the game where a party has to respond to an offer, he also has the additional alternative of withdrawing from the negotiations, thereby enforcing the outside option point  $e$ . The next proposition (Binmore, 1985; Shaked and Sutton, 1984) asserts that the inclusion of the outside option affects the equilibrium outcome of the strategic models only if at least one of the parties prefers the outcome  $e$  to one of the agreements  $x^*, y^* \in X$ , which characterize the equilibrium in the absence of the outside options.

*Proposition 6.* (a) Under the assumptions of Proposition 2, if  $(x^*, 0) \succeq_i (e, 0)$  and  $(y^*, 0) \succeq_i (e, 0)$ ,  $i = 1, 2$ , then the unique perfect equilibrium of the game with the time-preference model with the added outside options is the one described in Proposition 2. (b) Under the assumptions of Proposition 3, if  $x^* \succeq_i e$  and  $y^* \succeq_i e$ ,  $i = 1, 2$ , then the unique perfect equilibrium of the game with the exogenous risk model with the added outside options is the one described in Proposition 3.

Proposition 6 leads to the conclusion that  $s^0$  should not be identified with the outside option point  $e$ . Rather, despite the availability of these options, it remains appropriate to identify  $s^0$  with one of the outcomes  $g$  or  $b$ , depending on the nature of the modelled situation. The presence of  $e$  just places restrictions on the solution, which is now the argument of  $\max (s_1 - s_1^0)(s_2 - s_2^0)$  subject to  $s_i \geq u_i(e)$ ,  $i = 1, 2$ .

To recognize the significance of this result, the reader should realize that the outcome  $e$  might be very different from  $g$  or  $b$ , and hence the prediction of the model could

be substantially different according to whether  $s^0$  is identified with  $e$  or  $g$ . For example, in the case of wage negotiations,  $e$  is identified with the income streams that the employee can get on another job and that the employer can derive from hiring a different worker. These streams could be quite different from the streams of income accruing to the parties in the course of the dispute, which are identified with  $g$ . To observe the potential difference between  $e$  and  $b$ , consider another example in which the two parties bargain over the gains from a business opportunity that they hope to exploit together. Suppose that they can exploit this opportunity separately, in which case their combined gains will be lower than the gains of a joint venture. Suppose further that, as long as they do not exploit the opportunity, there is a risk that it will be snatched by a third party and be lost for them. The possibility of separate exploitation is identified with  $e$ , while the potentially very different possibility of loss is identified with  $b$ .

□ **Symmetry and asymmetry.** The symmetry axiom of the Nash bargaining solution requires that if  $(n_1, n_2)$  is the solution of  $(S, s^0)$ , then  $(n_2, n_1)$  is the solution for the symmetrically permuted problem. This axiom is often interpreted as stating that all relevant differences between the parties are already captured by  $(S, s^0)$ , and therefore an asymmetry in the solution can reflect only asymmetries in  $(S, s^0)$ .

In the strategic models of Section 2 there are two types of asymmetry. First, there is a slight procedural asymmetry, which gives the advantage to the party who makes the first proposal. Second, the parties differ with respect to their preferences (time preferences or attitudes towards risk) and with respect to their disagreement points ( $g_i$  or  $b_i$ ). Asymmetries of the second type are built into the static representations  $(S, s^0)$  derived in Section 3. The asymmetry in the procedure essentially disappears when the length  $\Delta$  of a single bargaining period tends to zero. Therefore, it is not surprising that the limiting equilibrium outcomes of the strategic models are symmetric solutions for the corresponding pairs  $(S, s^0)$ .

The static bargaining theory also considers asymmetric solutions. The set of asymmetric Nash solutions is obtained when the symmetry axiom is deleted from the axiomatization of the Nash solution (Roth, 1979). Each of these solutions is characterized by the maximization problem  $\max_{s \in S} (s_1 - s_1^0)^\alpha (s_2 - s_2^0)^{1-\alpha}$  for some  $\alpha$ ,  $0 \leq \alpha \leq 1$ . Modellers often use the asymmetric Nash solution in an attempt to capture some imprecisely defined differences in "bargaining power," where a large exponent  $\alpha$  is interpreted as representing a relatively high bargaining power of party 1. The analysis of Sections 2 and 3 provides a more solid grounding for the modelling decisions involved in the applications of asymmetric solutions. First, note that there are several sources of asymmetry in bargaining power. These include the above-mentioned asymmetries in preferences, disagreement points, and the bargaining procedure; and perhaps there are also asymmetries in the parties' beliefs about some determinants of the environment. As we have seen, asymmetries in the preferences<sup>3</sup> and disagreement points are already captured in the construction of  $(S, s^0)$ . It follows that one need not use the asymmetric solutions unless the modelled situations display the remaining asymmetries.

In what follows we shall demonstrate how the power weights  $\alpha$  and  $1 - \alpha$  can be chosen to reflect some possible asymmetries in the procedure and in the parties' beliefs. First, consider the strategic models of Section 2, and let  $\Delta_i$  be the length of the interval that elapses between  $i$ 's reaction to  $j$ 's proposal and the next point at which  $i$  proposes to  $j$ . Throughout the analysis we assumed that the procedure was symmetric in the sense that  $\Delta_1 = \Delta_2$ . We

<sup>3</sup> Since the time preferences  $\succeq_i$  are represented by  $\delta^t u_i(x)$  with the same notional discount factor  $\delta$  for both parties, all the differences between the parties' time preferences are already captured by  $u_1$  and  $u_2$ . If different discount factors are used, then  $u_1$  and  $u_2$  do not capture all the differences, and the resulting asymmetry must be treated by using an asymmetric Nash solution.

can, however, consider an asymmetric procedure in which  $\Delta_1 \neq \Delta_2$ . Upon computing the unique perfect equilibrium for each of the strategic models, and letting  $\Delta_1$  and  $\Delta_2$  approach zero while keeping their ratio constant, it is easy to verify that the limiting equilibrium outcomes of both models coincide with the respective asymmetric solutions with poor  $\alpha = \Delta_2/(\Delta_1 + \Delta_2)$ . That is, in each of the models the limiting equilibrium outcome is the solution of  $\max_{s \in S} (s_1 - s_1^0)^\alpha (s_2 - s_2^0)^{1-\alpha}$ , where  $(S, s^0)$  is the static representation that corresponds to that model as described in Section 3. Note that the larger  $\Delta_2$  is relative to  $\Delta_1$ , the larger  $\alpha$  will be, and hence the “stronger” is party 1. Similar asymmetric solutions arise if the proposer at each time  $t\Delta$  is chosen with different probabilities for the two players.

Next consider the bargaining model with a risk of breakdown of negotiations. Suppose that  $\Delta_1 = \Delta_2 = \Delta$  but the parties differ in their beliefs concerning the likelihood of a breakdown. Let  $p_i(\Delta) = 1 - e^{-\lambda_i \Delta}$  be the probability assigned by party  $i$  to the event that the process will break down during a single bargaining period. As  $\Delta$  approaches zero, the unique perfect equilibrium of the model with asymmetric beliefs approaches the solution for  $\max_{s \in S} (s_1 - s_1^0)^\alpha (s_2 - s_2^0)^{1-\alpha}$ , where  $\alpha = \lambda_2/(\lambda_1 + \lambda_2)$  and  $(S, s^0)$  is the static representation described in Section 3’s discussion of the exogenous-risk model. Thus, the higher is party  $i$ ’s estimate of the probability of breakdown, the lower is his bargaining power.

Note that this last possibility involves an “agreement to disagree.” This may often be a reasonable behavioral assumption, but is not consistent with common knowledge (Aumann, 1976).

## 5. Conclusion

■ In this article we have shown how some of the data of an economic situation that involves bargaining can be used to apply Nash’s bargaining solution to the problem. The main idea has been to use the insights of the strategic approach to bargaining in making the modelling judgments involved in the selection of the utility representations and the disagreement point for the application of Nash’s solution. We have demonstrated this method for two important types of environments in which the incentives to reach an agreement were, respectively, the parties’ impatience and the parties’ fear of breakdown of the negotiations. Furthermore, although the article does not deal with environments in which the two types of motives are present together, it is straightforward to conclude from our work how the appropriate Nash problem is to be constructed in such cases.

## Appendix

■ The proof of Proposition 4 follows.

*Proof of Proposition 4.* Let  $(u_1^N, u_2^N)$  denote the Nash solution expressed in utility terms; that is,

$$(u_1^N, u_2^N) = (u_1(x_N^{TP}), u_2(x_N^{TP})).$$

It is well known that  $(u_1^N, u_2^N)$  is characterized by the following two equations:

$$u_2 = \psi(u_1^N) \tag{A1}$$

$$|\psi_-(u_1^N)| \leq u_2^N/u_1^N \leq |\psi_+(u_1^N)|, \tag{A2}$$

where  $\psi_-$  and  $\psi_+$  are the left-hand side and right-hand side derivatives of  $\psi$ .

Thus, to prove the proposition we have to show that  $\lim_{\Delta \rightarrow 0} [u_1(x^*(\Delta)), u_2(x^*(\Delta))] = (u_1^N, u_2^N)$ .

From Proposition 2 we have

$$u_1(y^*(\Delta)) = \delta^\Delta u_1(x^*(\Delta)) \tag{A3}$$

$$u_2(x^*(\Delta)) = \delta^\Delta u_2(y^*(\Delta)). \tag{A4}$$

For brevity, let us denote  $u_i(x^*(\Delta))$  by  $u_i^\Delta$ . Since  $u_2 = \psi(u_1)$ , it follows from (A3) and (A4) that

$$\psi(u_1^\Delta) = \delta^\Delta \psi(\delta^\Delta u_1^\Delta). \tag{A5}$$

Since the range of  $u_1$  is compact, it is sufficient to show that, for every sequence  $\Delta(n) \rightarrow 0$  such that  $u_1^{\Delta(n)}$  is convergent,  $u_1^{\Delta(n)} \rightarrow u_1^N$ . Thus, let  $\Delta(n) \rightarrow 0$  such that  $u_1^{\Delta(n)} \rightarrow \bar{u}_1$ . It will be sufficient to show that  $\bar{u}_1$  and  $\psi(\bar{u}_1)$  satisfy (A1) and (A2). It follows from the fact that  $\psi$  is concave and decreasing that

$$|\psi'(\bar{u}_1)| \leq \lim_{n \rightarrow \infty} \left| \frac{\psi(u_1^{\Delta(n)}) - \psi(\delta^{\Delta(n)} u_1^{\Delta(n)})}{u_1^{\Delta(n)} - \delta^{\Delta(n)} u_1^{\Delta(n)}} \right| \leq |\psi'(\bar{u}_1)|. \quad (\text{A6})$$

From (A5) we have that the limit of the central term of (A6) is

$$\lim_{n \rightarrow \infty} \left| \frac{(\delta^{\Delta(n)} - 1)\psi(\delta^{\Delta(n)} u_1^{\Delta(n)})}{(1 - \delta^{\Delta(n)})u_1^{\Delta(n)}} \right| = \frac{\psi(\bar{u}_1)}{\bar{u}_1}.$$

Thus,  $\bar{u}_1$  and  $\psi(\bar{u}_1)$  satisfy (A1) and (A2), and, therefore,  $\bar{u}_1 = u_1^N$ ,  $\psi(\bar{u}_1) = u_2^N$ . *Q.E.D.*

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