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Aggregation of Equivalence Relations

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Abstract: Each of n attributes partitions a set of items into equivalence classes. A *consistent aggregator* of the n partitions is defined as an aggregate partition that satisfies an independence condition and a unanimity condition. It is shown that the class of consistent aggregators is precisely the class of *conjunctive aggregators*. That is, for each consistent aggregator there is a nonempty subset N of the attributes such that two items are equivalent in the aggregate partition if and only if they are equivalent with respect to each attribute in N.

Keywords: Classification; aggregation; consistency; conjunctive aggregator.

1. Introduction

Classification methods are often based on similarities within attributes of the items to be classified. Each attribute, such as number of flower petals, mammalian reproductive type, or sex, can be viewed as a set of values or nominal designations. We say that two items are *equivalent* with respect to an attribute if they have the same value of that attribute. When a specified set of items is compared over a number of attributes, each attribute partitions the set into equivalence classes according to the attribute's values. The problem of classification is then to say how these attribute-specific partitions are to be merged or aggregated into a holistic set of equivalence classes.

This paper considers the effect of restrictions on the aggregation of attribute-specific partitions in arriving at a holistic classification scheme. In particular, we show that two appealing consistency conditions for aggregation dramatically limit classification schemes based on attribute equivalence. Our

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methodology stems from social choice theory (Arrow 1951, Fishburn 1973) for the aggregation of individuals' preferences into social preference relations and from its generalization to other forms of aggregation developed by Wilson (1975) and Rubinstein and Fishburn (1985).

The paper's main result says that if an aggregator over n attributes satisfies an independence condition and a unanimity condition, then it must be a conjunctive aggregator. This means that there is a nonempty subset of the n attributes such that two items are holistically equivalent — as determined by aggregation — if and only if they are equivalent on each attribute in the designated subset. Since there are $2^n - 1$ nonempty subsets of n attributes, the two consistency conditions only limit the aggregator to one of $2^n - 1$ possible conjunctive forms.

The next section presents our formulation in terms of the aggregation of equivalence relations, which are tantamount to partitions of sets of items. The main result and its proof appear in the third section. The paper concludes with a brief summary.

2. Formulation

Let A be a finite set. We view A as an arbitrary set of items for classification in the sense that their values for the attributes under consideration are not predetermined but can range over all possible combinations of those values.

A binary relation $\sim$ on A is defined to be an *equivalence relation* if, for all $a, b, c \in A$,

$$a \sim a \quad \text{(reflexivity)}$$

$$a \sim b \Rightarrow b \sim a \quad \text{(symmetry)}$$

$$(a \sim b,\ b \sim c) \Rightarrow a \sim c \quad \text{(transitivity).}$$

Each equivalence relation on A is equivalent to a partition of A into mutually disjoint subsets, called *equivalence classes*, such that a and b are in the same equivalence class if and only if $a \sim b$.

Let n be the number of attributes under consideration, and for each $i \in \{1, 2, \ldots, n\}$ let $\sim_i$ denote an equivalence relation on A that might result from a partition of A into equivalence classes based on attribute i. An *aggregator* is defined as a function F that assigns a holistic equivalence relation $F(\sim_1, \ldots, \sim_n)$ on A to each possible n-tuple $(\sim_1, \ldots, \sim_n)$ of equivalence relations on A. Thus, no matter what $\sim_1$ through $\sim_n$ turn out to be, F provides a summary classification by the equivalence classes in

$F(\sim_1, \ldots, \sim_n)$. For notational convenience we designate $F(\sim_1, \ldots, \sim_n)$ as $\sim$. Similarly, $\sim'$ stands for $F(\sim_1', \ldots, \sim_n')$.

An aggregator F is said to be *consistent* if it satisfies the following two conditions:

C1. For all $a, b \in A$ and all n-tuples $(\sim_1, \ldots, \sim_n)$ and $(\sim_1', \ldots, \sim_n')$ of equivalence relations on A, if $a \sim_i b <=> a \sim_i' b$ for $i = 1, \ldots, n$, then $a \sim b <=> a \sim' b$;

C2. For all $a, b \in A$ and all $(\sim_1, \ldots, \sim_n)$, if $a \sim_i b$ for $i = 1, \ldots, n$ then $a \sim b$, and if $a \not\sim_i b$ for $i = 1, \ldots, n$ then $a \not\sim b$. [$a \not\sim_i b$ means not $(a \sim_i b)$; $a \not\sim b$ means not $(a \sim b)$.]

C1 is an independence condition which says that the aggregate relation between a and b depends only on the attribute relations between those two items. For example, if $n = 3$ and if $a \sim_1 b$, $a \sim_2 b$, $a \not\sim_3 b$ and $a \sim_1' b$, $a \sim_2' b$, $a \not\sim_3' b$, then a and b are holistically equivalent in the first case if and only if they are holistically equivalent in the second.

C2 is a unanimity, constancy, or faithfulness condition which says that if two items are equivalent on every attribute then they are holistically equivalent. Moreover, if they are not equivalent on any attribute, then they are not holistically equivalent.

Our concern with consistent aggregators leads to another class of aggregators known as conjunctive aggregators. We say that aggregator F is *conjunctive* if there is a nonempty $N \subseteq \{1, \ldots, n\}$ such that, for all $a, b \in A$ and all n-tuples $(\sim_1, \ldots, \sim_n)$ of equivalence relations on A,

$$a \sim b \text{ if and only if } a \sim_i b \text{ for every } i \in N .$$

Thus a conjunctive aggregator bases holistic equivalence on unanimous equivalence within a nonempty subset of attributes. There are clearly $2^n - 1$ different conjunctive aggregators for n attributes.

3. The Result

Theorem. *Suppose A has at least three elements. Then the set of consistent aggregators equals the set of conjunctive aggregators.*

Proof. It is easily seen that every conjunctive aggregator is consistent. To prove the converse, assume that F is consistent. We show that it is also conjunctive.

Given $(\sim_1, \ldots, \sim_n)$ and $a \neq b$ in A, let

$$v_i = 1 <=> a \sim_i b$$

$$v_i = 0 <=> a \not\sim_i b \ .$$

The n-tuple $(v_1, \ldots, v_n)$ describes how a and b are related on the n attributes. By C1, $a \sim b$ or $a \not\sim b$ depends solely on $(v_1, \ldots, v_n)$ as $(\sim_1, \ldots, \sim_n)$ ranges over all possibilities. We can therefore define a function F_{ab} on the set of all n-tuples of 0's and 1's that describes the behavior of F for a versus b as follows:

$$F_{ab}(v_1, \ldots, v_n) = 1 \iff a \sim b$$

$$F_{ab}(v_1, \ldots, v_n) = 0 \iff a \not\sim b \ .$$

Clearly, $F_{ba} = F_{ab}$. We also claim that $F_{ab} = F_{bc}$ when a, b and c are distinct items in A. To verify this, note first that for every $(v_1, \ldots, v_n)$ in $\{0,1\}^n$ there is an n-tuple $(\sim_1, \ldots, \sim_n)$ of equivalence relations on A such that

$$a \sim_i b \text{ and } b \sim_i c \quad \text{if} \quad v_i = 1$$

$$a \not\sim_i b \text{ and } b \not\sim_i c \quad \text{if} \quad v_i = 0$$

and $a \sim_i c$ for all i. By C2, $a \sim c$. Then, by the transitivity of $\sim$, $a \sim b \iff b \sim c$. Hence $F_{ab} = F_{bc}$.

Since A is presumed to be finite, it follows from repeated applications of these results that $F_{ab} = F_{cd}$ for all $a \neq b$ and $c \neq d$ in A. Let $G = F_{ab}$ for all $a \neq b$.

Now for every $N \subseteq \{1, \ldots, n\}$ let 1_N be the vector in $\{0,1\}^n$ that has 1 in position i if $i \in N$ and 0 otherwise. We prove next that, for all subsets N and M of $\{1, \ldots, n\}$,

$$G(1_N) = G(1_M) = 1 \quad \Rightarrow \quad G(1_{N \cap M}) = 1.$$

Let a, b and c be distinct elements in A. Then there is an n-tuple $(\sim_1, \ldots, \sim_n)$ of equivalence relations on A in which

$$a \sim_i b \iff i \in N$$

$$b \sim_i c \iff i \in M$$

$$a \sim_i c \iff i \in N \cap M.$$

Suppose $G(1_N) = G(1_M) = 1$, i.e., $a \sim b$ and $b \sim c$. Then $a \sim c$ by transitivity, so $1 = F_{ac}(1_{N \cap M}) = G(1_{N \cap M})$.

Finally, let N be a minimal subset of $\{1, \ldots, n\}$ for which $G(1_N) = 1$. By C2, N is nonempty. Moreover, by the result of the preceding paragraph, for all $M \subseteq \{1, \ldots, n\}$, $G(1_M) = 1$ if and only if $N \subseteq M$. It follows that F is the conjunctive aggregator based on the attributes in N.

4. Summary

Our theorem says that if an aggregator of n equivalence relations on three or more items is consistent according to C1 and C2 regardless of the specific forms taken by those relations, then the aggregate relation $\sim$ is completely determined by a nonempty subset N of $\{1, \ldots, n\}$ by

$$a \sim b \iff (a \sim_i b \text{ for all } i \in N) \ .$$

This conjunctive form says that all i not in N are irrelevant to the aggregate classification while every i in N is essential and equally weighted. The theorem does not say anything about which i are essential.

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