Reprinted from JOURNAL OF ECONOMIC THEORY

Vol. 38, No. 1, February 1986
Printed in Belgium

Algebraic Aggregation Theory*

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Received July 10, 1984; revised June 21, 1985

An aggregation procedure merges a list of objects into a representative object. This paper considers the problem of aggregating n rows in an n-by-m matrix into a summary row, where every entry is an element in an algebraic field. It focuses on consistent aggregators, which require each entry in the summary row to depend only on its column entries in the matrix and to be the same as the column entry if the column is constant. Consistent aggregators are related to additive, linear and projective aggregators. *Journal of Economic Literature* Classification numbers: 025, 213. © 1986 Academic Press, Inc.

1. INTRODUCTION

Many aggregation problems in economics and other areas share common structural features. This paper explores an algebraic formulation for aggregation that captures much of this commonality and offers a unifying framework for the analysis of diverse aggregation problems.

Following the appearance of Arrow's impossibility theorem [1], social choice theory has studied logical restrictions on ways to aggregate individuals' preferences into social preferences [4]. Robert Wilson, in a remarkable but largely ignored paper [7], pioneered the extension of the social-choice approach into other areas by asking "whether procedures for aggregating attributes other than preferences are subject to similar restric-

*Ariel Rubinstein's research was done partly under the sponsorship of The Institute of Advanced Studies at The Hebrew University. The authors thank Kim Border for permission to cite his result on probability aggregation, and Ed Green for showing them how to use their results to prove Border's theorem. We also thank an anonymous referee for very helpful remarks and suggestions.

0022-0531/86 $3.00

TABLE I

i	j			
	1	2	$\cdots$	m
1	x_{11}	x_{12}	$\cdots$	x_{1m}
2	x_{21}	x_{22}		x_{2m}
3	x_{31}	x_{32}		x_{3m}
$\vdots$	$\vdots$			
n	x_{n1}	x_{n2}		x_{nm}
	$f_1(\,)$	$f_2(\,)$		$f_m(\,)$

tions" (p. 89). The current paper renews Wilson's concern. We believe that its main contribution is the adoption of a unifying *algebraic* framework for the theory of aggregation.

We consider the situation illustrated in Table I. Each element in the n-by-m matrix is contained in an algebraic field B, and each row is a vector $x_i = (x_{i1}, ..., x_{im})$ in a designated subset X of the vector space B^m over the field B. An *aggregator* f is a mapping from X^n into X:

$$f(x_1, ..., x_n) = (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n)).$$

We consider the real field $\mathbb{R}$ and finite (Galois) fields for B and will focus on so-called consistent aggregators. These require each f_j to depend only on the elements in column j and to have value b when every element in column j is b.

We stress that a great variety of aggregation problems can be modeled within this algebraic framework. The parameters n, m, B, and X and the interpretations of the rows and columns in the matrices to be aggregated are subject to choice. We illustrate possible choices by descriptions of five problems that include the requirements imposed on consistent aggregators. Table II shows how these problems fit into our formal model. $GF(2)$ is the Galois field of order 2 where $B = \{0, 1\}$.

PROBLEM 1. Each of n experts assigns a probability distribution over m states. The aggregator is to assign a probability distribution over the m states that depends on the individuals' distributions. The aggregate probability for each event is to depend only on the individuals' probabilities for that event.

The same formulation applies to aggregation problems in which a single person numerically rates each of m objects against each of n criteria.

TABLE II

Formulation of Five Aggregation Problems

Problem # and type	n	m	B	X in B^m
1: Probabilities	Experts or individuals	Events	$\mathbb{R}$	$\{(x^1,..., x^m): x^j \geqslant 0$ and $\sum x^j = 1\}$
2: Probabilities	Individuals	Event ratios j over $j+1 \pmod m$	$\mathbb{R}$	$\{(x^1,..., x^m): x^j > 0$ and $\prod x^j = 1\}$
3: Voting possibilities	Voters	Candidates	$GF(2)$	$\{(0,..., 0, 1_j, 0,..., 0): 1 \leqslant j \leqslant m\}$
4: Orderings	Voters	Ordered pairs of candidates	$GF(2)$	$\{(x_{hk})_{1 \leqslant h<k \leqslant K}$ in $\{0,1\}^{\binom{K}{2}}$: $>$ with $h>k$ if $x_{hk}=1$, $k>h$ if $x_{hk}=0$, is an ordering$\}$
5: Equivalence relations	Equivalence criteria	Pairs of items	$GF(2)$	$\{(x_{hk})_{1 \leqslant h<k \leqslant K}$ in $\{0,1\}^{\binom{K}{2}}$: $\approx$ with $h \approx k$ iff $x_{hk}=1$ is an equivalence relation$\}$

PROBLEM 2. Probabilities are to be aggregated as in Problem 1, and probability 0 is never allowed in a distribution. Unlike Problem 1, we now require that the ratio of the aggregate probabilities of any two events shall depend only on the individuals' probability ratios for those two events.

PROBLEM 3. Each of n voters is to vote for one of m nominated candidates. The determination of each candidate as the winner or a loser is to depend only on the voters' reactions to that candidate.

PROBLEM 4. Each of n individuals is to submit a total ordering of K social alternatives—a typical assumption in some social choice problems. The aggregator is to provide a social ordering of the K alternatives that satisfies Arrow's binary independence condition (social preference between two alternatives depends only on the individuals' preferences between those two) and Pareto unanimity (if every individual prefers a to b, then a is socially preferred to b).

A similar formulation applies to non-voting situations. An individual-tastes example of the type considered by May [6] assumes that a person is to rank songs according to her tastes. She may simply say that s_1 ranks ahead of s_2 because she likes s_1 more. However, such a preference may be based on more primitive factors such as originality, rhythm, emotional content, and chord structures, so her holistic ranking can be viewed as an aggregation of other relations.

PROBLEM 5. In a classification situation, a population of K items, such as flowers, firms, or football teams, is to be partitioned into subsets of similar items. The classification may be based on more primitive equivalence relations, each of which corresponds to a row of the matrix. For example, rows for flower classifications might refer to color, number of petals, growing season, and so forth. The aggregation combines the primitive relations into a global equivalence relation. The global equivalence of any two items is to depend only on the primitive relations between those two.

These descriptions illustrate a few of the situations in which algebraic aggregation theory is applicable. We comment on them further in later sections. The next section says a little more about the algebraic orientation of our approach, then defines consistent aggregators and other types of aggregators that are used in our analysis. The third section presents basic aggregation theorems for special X structures.

2. AGGREGATORS

Throughout, n and m are positive integers greater than or equal to 2, and B is a field (see [5] for background on finite fields). We view B^m as a vector space over B: with $(x^1,..., x^m), (y^1,..., y^m) \in B^m$ and $b \in B$, $(x^1,..., x^m)+(y^1,..., y^m)=(x^1+y^1,..., x^m+y^m)$ and $b(x^1,..., x^m)=(bx^1,..., bx^m)$. We say that $Y \subseteq B^m$ is a *linear subspace* of B^m if the restriction of B^m to Y is a vector space over B (closed under the same addition and multiplication operations as for B^m). A *translate* of a linear subspace Y of B^m is a set of the form $Y+x=\{y+x: y \in Y\}$ with $x \in B^m$. In addition, $Y \subseteq B^m$ is a *hyperplane* if there are $b_j \in B$ for $j=1,..., m$ and $b \in B$ such that

$$Y=\left\{y \in B^m: \sum_{j=1}^{m} b_j y^j = b\right\}.$$

A hyperplane in B^m is a translate of a linear subspace, for if $Y=\{y \in B^m: \sum b_j y^j = b\}$ then any set of the form $Y-x$ for fixed $x \in Y$ is a linear subspace of B^m. On the other hand, a linear subspace need not be a hyperplane.

We always assume that $X \subseteq B^m$ with $|X| \geqslant 2$. Vectors in X will be written as x, x_i, and $(x_{i1},..., x_{im})$. A constant vector $(a,..., a)$ in B^m or X will sometimes be denoted as **a**. Similar symbols for *column* vectors in B^n will be used in some proofs, with appropriate warning.

An *aggregator* is any mapping $f: X^n \to X$. The set of all aggregators is denoted by F. A key aspect of this definition, which could be relaxed in a more general approach, is that $f(x_1,..., x_n)$ must be in the same "feasible set" X as each of its arguments. The jth component of $f(x_1,...,x_n)$ will sometimes be denoted by $f_j(x_1,..., x_n)$, so

$$f(x_1,..., x_n) = (f_1(x_1,..., x_n),..., f_m(x_1,..., x_n)).$$

Depending on the setting and purpose of aggregation, it may be natural to impose certain constraints on aggregators. Two such constraints, applied to all $(x_1,..., x_n)$ and $(x'_1,..., x'_n)$ in X^n and all j in $\{1,..., m\}$, are

C_1. $(x_{1j},..., x_{nj}) = (x'_{1j},..., x'_{nj}) \Rightarrow f_j(x_1,..., x_n) = f_j(x'_1,..., x'_n)$

C_2. $(x_{1j},..., x_{nj}) = (b,..., b) \Rightarrow f_j(x_1,..., x_n) = b.$

Condition C_1 is an independence condition which says that the jth component of the aggregator shall depend only on the jth column of the aggregation matrix. In preference aggregation, C_1 might indicate that the group preference between candidates c_1 and c_2 shall depend only on the individuals' preferences between those two candidates. In straightforward subjective probability aggregation, C_1 says that the consensus probability for state j depends only on the individuals' subjective probabilities for that state.

Condition C_2 is a unanimity, constancy, or faithfulness condition. If everyone prefers c_1 to c_2, then c_1 is socially preferred to c_2; if everyone assigns probability 0.4 to state j, then the consensus probability for state j is 0.4; if two flowers are equivalent for every classification criterion, then they are holistically equivalent.

Each column interpretation in the preceding two paragraphs is only meant to illustrate one way of identifying columns in each context. In the voting context or the probability context, columns could be interpreted differently with corresponding and different interpretations of C_1 and C_2.

We refer to aggregators that satisfy C_1 and C_2 as *consistent aggregators* and let

$$F_C = \{f \in F: f \text{ satisfies } C_1 \text{ and } C_2\}.$$

We regard F_C as a preeminent class of aggregators and will focus on it in the present work. Since Arrow's impossibility theorem and related results tell us that consistent aggregators may have initially unsuspected properties, we consider their relationships to other classes of aggregators. Some of these are useful and interesting in their own right.

One such class is F_A, the set of *additive aggregators*. With X_j the projection of X on its jth coordinate,

$$F_A = \{f \in F: f \text{ satisfies } C_1, \text{ and } f_j(y+z) = f_j(y) + f_j(z) \text{ for all } j \leqslant m \text{ and all column vectors } y \text{ and } z \text{ for which } y, z, y+z \in X_j^n\}.$$

A closely related class is the set F_S of normalized *linear aggregators*:

$$F_S = \{f \in F: \text{there exist } \lambda_1,\ldots,\lambda_n \in B \text{ such that } \Sigma\lambda_i = 1 \text{ and, for all } (x_1,\ldots,x_n) \in X^n, f(x_1,\ldots,x_n) = \Sigma\lambda_i x_i\}.$$

It is easily shown that if X is a translate of a linear subspace of B^m then every function $f(x_1,\ldots,x_n) = \Sigma\lambda_i x_i$ with $\Sigma\lambda_i = 1$ is in F_S.

Our final general class of aggregators is the set F_P of *projection aggregators* (dictatorial aggregators):

$$F_P = \{f \in F: \text{there is an } i \in \{1,\ldots,n\} \text{ such that, for all } (x_1,\ldots,x_n) \in X^n, f(x_1,\ldots,x_n) = x_i\}.$$

Note that $F_P \subseteq F_S \subseteq F_A$.

3. Hyperplanes and Linear Subspaces

This section presents a general theorem for consistent aggregators on hyperplanes, then considers ramifications of the theorem and related results. Specific applications are discussed in the next section.

THEOREM 1. *Suppose* $m \geqslant 3$ *and* $X = \{(x^1,\ldots,x^m) \in B^m: \Sigma_j b_j x^j = b\}$ *with* $b_j \neq 0$ *for all* $j \leqslant m$. *Then* $F_C \subseteq F_A$.

Remark 1. As shown by the following proof, the theorem is true also if $b_j \neq 0$ for at least three j if the projection X_j is a singleton for each j for which $b_j = 0$.

Proof. Throughout the proof of Theorem 1, y and z denote n-element column vectors in B^n, and $\mathbf{b}$ denotes the n-vector with b as each component. Assume the hypotheses of the theorem along with $f \in F_C$. We are to show that $f \in F_A$. It suffices to work with $m = 3$ since extension to larger m is straightforward.

Consider first the aggregation matrices M_1 and M_2:

$$M_1 = (y, \mathbf{0}, t)$$

$$M_2 = \left(\mathbf{0}, \frac{b_1}{b_2} y, t\right) \qquad \text{with } t = (\mathbf{b} - b_1 y)/b_3.$$

Since $f(M_1)$ and $f(M_2)$ are in X,

$$b_1 f_1(y) + 0 + b_3 f_3(t) = b,$$

$$0 + b_2 f_2\left(\frac{b_1}{b_2} y\right) + b_3 f_3(t) = b,$$

and therefore $b_1 f_1(y) = b_2 f_2((b_1/b_2)\, y)$.

Let $w = [\mathbf{b} - b_1(y+z)]/b_3$. Then M_3 and M_4 are in X when

$$M_3 = (y+z, \mathbf{0}, w) \qquad \text{and} \qquad M_4 = (z, (b_1/b_2)\, y, w).$$

Since $f(M_3), f(M_4) \in X$,

$$b_1 f_1(y+z) = b - b_3 f_3(w)$$

$$b_1 f_1(z) + b_2 f_2((b_1/b_2) y) = b - b_3 f_3(w),$$

so that

$$b_1 f_1(y+z) = b_1 f_1(z) + b_2 f_2((b_1/b_2) y) = b_1 f_1(z) + b_1 f_1(y).$$

Thus $f_1(y+z) = f_1(y) + f_1(z)$. A similar proof applies to $j > 1$. ∎

Our first corollary gives a bounded version of Theorem 1 when $B = \mathbb{R}$. Because of its relevance to applications, we focus on nonnegativity.

Corollary 1. *Suppose* $m \geqslant 3$, $X = \{(x^1, \dots, x^m) \in \mathbb{R}^m \colon \Sigma b_j x^j = b$ *and* $x^j \geqslant 0$ *for all* $j\}$ *with* b *and all* b_j *positive. Then* $F_C \subseteq F_A$.

The proof is a repetition of the preceding proof with nonnegativity checks.

The next corollary provides a condition that forces all aggregators to be in F_S as well as in F_A.

Corollary 2. *Given the hypothesis of Theorem* 1, *let* f *be in* F_C. *Then* $f \in F_S$ *if* B *is a finite field, or if* $B = \mathbb{R}$ *and every* f_j *is continuous or monotone.*

Proof. Assume the hypotheses of Theorem 1 along with $f \in F_C$, so that $f \in F_A$. If B is a finite field, let $\lambda_i = f_1(e_i')$, where e_i' is the column vector with 1 in position i and 0 elsewhere. By additivity for general y,

$$f_1(y)=f_1\left(y_1\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}+y_2\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}+\cdots+y_n\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}\right)=\Sigma y_i\lambda_i.$$

Then the generalization $b_jf_j(y)=b_kf_k((b_j/b_k)y)$ of the similar result for $(j,k)=(1,2)$ in the proof of Theorem 1 implies $f_j(y)=\Sigma\lambda_i y_i$ for every j. If $B=\mathbb{R}$ then, as in the finite case, we get $f_1(y)=\Sigma\lambda_i y_i$ for integral y, then $f_1(y)=\Sigma\lambda_i y_i$ for a rational subset of $\mathbb{R}^n$ that is dense in $\mathbb{R}^n$. Either monotonicity or continuity implies that the linear form holds for all $y\in\mathbb{R}^n$. Finally, given $a\in B$, $a\neq 0$, since $a=f_1(\mathbf{a})=\Sigma\lambda_i a$, $\Sigma\lambda_i=1$ and therefore $f\in F_S$. ∎

The restriction of $m\geqslant 3$ (or $b_j\neq 0$ for at least three j) is important in Theorem 1, since otherwise the theorem can fail. Consider $m=2$, $B=\{0,1\}$, and $b_1=b_2=b=1$, so $X=\{(1,0),(0,1)\}$. Take $n=2$, let $f(x_1,x_2)=(1,0)$ when at least one x_i is $(1,0)$, and let $f(x_1,x_2)=(0,1)$ otherwise. Then f is consistent, but not additive, since additivity would give

$$0=1+1=f_1\begin{pmatrix}0\\1\end{pmatrix}+f_1\begin{pmatrix}1\\0\end{pmatrix}=f_1\left(\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}\right)=f_1\begin{pmatrix}1\\1\end{pmatrix}=1.$$

We conclude this section with observations on the projective or dictatorial aggregators in F_P.

The following theorem says that if B is a finite field and X is not a translate of a linear subspace of B^m, then $F_P=F_C\cap F_S$, i.e., every linear consistent aggregator is dictatorial. The proof of this theorem demonstrates the algebraic approach's potential in the theory of aggregation.

THEOREM 2. *If B is a finite field and $F_C\cap F_S\neq F_P$, then X is a translate of a linear subspace of B^m.*

Proof. Let B be a finite field and suppose that $f\in(F_C\cap F_S)\backslash F_P$ with $f(x_1,\ldots,x_n)=\Sigma\lambda_i x_i$, where $\Sigma\lambda_i=1$. With $x\neq\mathbf{0}$ fixed in X and $Y=X-x$, we claim that Y is a linear subspace of B^m, and therefore need to prove that it is closed under addition and under multiplication.

Assume that $y_1,\ldots,y_n$ are in Y. Then $y_i+x\in X$ for all i, and therefore $\Sigma\lambda_i(y_i+x)=\Sigma\lambda_i y_i+x$ is in X and $\Sigma\lambda_i y_i$ is in Y. Since $\mathbf{0}\in Y$, it follows that $y\in Y\Rightarrow\lambda_i y\in Y$ for all i. If $\lambda_i\notin\{0,1\}$ for some i, then all scalar ($b\in B$) multiples of $y\in Y$ are in Y since $B\backslash\{0\}=\{\lambda_i,\lambda_i^2,\lambda_i^3,\ldots\}$ by virtue of the fact that the nonzero elements of a finite field B form a cyclic multiplicative group. Suppose $\lambda_i\in\{0,1\}$ for all i. Then, since $f\notin F_P$ by assumption, at least two $\lambda_i=1$, say $\lambda_1=\lambda_2=1$ for definiteness. Since y and $\mathbf{0}$ are in Y, $\lambda_1 y+\lambda_2 y=(\lambda_1+\lambda_2)y\in Y$. If $\lambda_1+\lambda_2\neq 0$, then, since $\lambda_1+\lambda_2\neq 1$, the preceding argument for $B\backslash\{0\}$ with $\lambda_1+\lambda_2$ in place of λ_i shows that Y is closed under multiplication. On the other hand, if $\lambda_1 + \lambda_2 = 0$, then $B = GF(2)$, and again we get closure under multiplication since $B = \{0, 1\}$.

It remains to prove that Y is closed under addition. Let λ_1 and λ_2 be nonzero λ_i, and let $\mu = \lambda_1 \lambda_2$. Clearly, $\mu \neq 0$, and there is some $K > 0$ such that $\mu^K = 1$, so $\mu^K y = y$ for all $y \in Y$. Since it follows from $(y \in Y \Rightarrow \lambda_1 y \in Y)$ in the preceding paragraph that $\lambda_1^{K-1} \lambda_2^K y$ and $\lambda_1^K \lambda_2^{K-1} z$ are in Y when $y, z \in Y$, we conclude that $y + z \in Y$ whenever $y, z \in Y$ since

$$y + z = \mu^K y + \mu^K z = \lambda_1(\lambda_1^{K-1} \lambda_2^K y) + \lambda_2(\lambda_1^K \lambda_2^{K-1} z).$$

Hence Y is closed under addition. ∎

4. Applications to Probability Aggregation

Although the preceding results apply to many different kinds of aggregation problems, we shall illustrate them with two examples of probability aggregation.

Suppose first that each of n experts is to be consulted for his or her probability distribution $p_i = (p_{i1}, \ldots, p_{im})$ over $m \geqslant 3$ mutually exclusive and exhaustive events. Their distributions are to be aggregated in such a way that the aggregation for event j depends only on the individuals' probabilities for event j. If the aggregator is to satisfy C_1 and C_2 also, then Corollary 1 with

$$X = \{(p^1, \ldots, p^m) : p^j \geqslant 0 \text{ and } \Sigma p^j = 1\}$$

implies that it is additive. If it is also required to be continuous, then Corollary 2 says that the aggregator is a weighted sum of the experts' probability vectors with nonnegative weights that sum to 1.

Our second example gives a simple proof of an unpublished result of Kim Border (see also [2]) that was suggested to us by Ed Green. Let

$$\Delta_0 = \{(p^1, \ldots, p^m) : p^j > 0 \text{ and } \Sigma p^j = 1\},$$

with all probabilities strictly positive. We apply the consistency conditions to ratios of probabilities rather than to single-event probabilities as in the preceding paragraph. For all $p_i = (p_{i1}, \ldots, p_{im})$ and $p'_i = (p'_{i1}, \ldots, p'_{im})$ in Δ_0 $(i = 1, \ldots, n)$, and all distinct j and k in $\{1, \ldots, m\}$, assume the following:

(C_1) if $p_{ij}/p_{ik} = p'_{ij}/p'_{ik}$ for all i, then the aggregate probability ratios for j and k are the same;

(C_2) if $p_{ij}/p_{ik} = b$ for all i, then the aggregate ratio is b.

Since $(p^1, \ldots, p^m) \in \Delta_0$ is determined by positive p^{j+1}/p^j for $j \leqslant m-1$ along with p^1/p^m—so $(p^2/p^1)\cdots(p^m/p^{m-1})(p^1/p^m)=1$—Theorem 1 can be directly applied to X defined as

$$X=\{(\log(p^2/p^1),..., \log(p^m/p^{m-1}), \log(p^1/p^m)): p\in\Delta_0\}$$
$$=\{(x^1,..., x^m)\in\mathbb{R}^m: \Sigma x^j=0\}.$$

Since (C_1) and (C_2) imply that the aggregator for X^n is in F_C, Theorem 1 implies that it is also in F_A, and if continuity is assumed also, then the aggregator is linear: symbolically, for all $j\leqslant m-1$,

$$\text{Agg}(\log(p^{j+1}/p^j))=\sum_i \lambda_i \log(p_{i,j+1}/p_{ij})$$
$$=\log\left[\prod_i (p_{i,j+1}/p_{ij})^{\lambda_i}\right],$$

and similarly for p^1/p^m. These imply that the aggregate probability for event j, based on $p_i=(p_{i1},..., p_{im})$ for $i\leqslant n$, is

$$\frac{\prod_i(p_{ij})^{\lambda_i}}{\sum_k\prod_i(p_{ik})^{\lambda_i}}.$$

5. Wilson Sets

Our final three sections take $B=GF(2)$ or, equivalently, $B=\mathbb{Z}_2$, the integers modulo 2. In B^m,

$$e_j=(0,..., 0, 1 \text{ (position } j), 0,..., 0)$$
$$1-e_j=(1,..., 1, 0 \text{ (position } j), 1,..., 1).$$

The present section focuses on $X\subseteq B^m$ that are suggested by Wilson's analysis [7]. Wilson does *not* assume that X (his attribute space) is multidimensional. Our use of $\{0, 1\}^m$ provides a concrete geometric interpretation for some of his ideas.

Our first observation is a direct implementation of Corollary 2.

Corollary 3. *Suppose* $m\geqslant 3$ *and* $X=\{(x^1,..., x^m): \sum x^j=b\}$ *with* $b\in\{0, 1\}$ *and* $X\subseteq\{0, 1\}^m$. *Then* $F_C=F_S$.

When $m=3$, the two X sets that satisfy the hypotheses of the corollary are

$(b=0)$ $\{(1, 1, 0), (1, 0, 1), (0, 1, 1), (0, 0, 0)\}$,

$(b=1)$ $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$.

In terms of edges of the unit cube, the set for $b=0$ consists of the three vertices adjacent to $(1, 1, 1)$, plus $(0, 0, 0)$, the complement of $(1, 1, 1)$. The set for $b=1$ consists of the three vertices adjacent to $(0, 0, 0)$, plus $(1, 1, 1)$, the complement of $(0, 0, 0)$. For general m, the $b=0$ and $b=1$ sets are composed of vectors with an even number of 1's and an odd number of 1's, respectively.

With Arrow as well as Wilson in mind, we now consider X that yield $F_C=F_P$, i.e., that force all consistent aggregators to be dictatorial. We know from Theorem 2 that if $F_C \subseteq F_S$ and X is *not* a translate of a linear subspace of B^m, then $F_C=F_P$. The W-sets defined in the next paragraph satisfy this condition on X (proof omitted).

We say that points in $\{0, 1\}^m$ are *adjacent* if they differ in exactly one coordinate, and are *complements* if their sum is 1. Given $X \subseteq \{0, 1\}^m$, X is a W_0*-set* if it consists of three or more points adjacent to a given vertex. In addition, X is a W_1*-set* if $m \geqslant 3$ and it contains every point adjacent to a given vertex and every point adjacent to the complement of the given vertex, but contains neither the given vertex nor its complement. If a W_0-set has $k<m$ points, then it will be constant ($|X_j|=1$) on all but k dimensions. It turns out that the dimensions on which it is constant can be ignored in the proof of our next theorem: see Remark 2 following the proof of the theorem. Hence, for now, we assume without loss of generality that a W_0-set consists of all points adjacent to a given vertex.

By complementary changes of variables, all W_0-sets are isomorphic to $\{e_1,..., e_m\}$, and all W_1-sets are isomorphic to a W_1-set that satisfies

$$\begin{aligned} \mathbf{0}, \mathbf{1} &\notin X, & \\ e_j &\in X & \text{for each } j, \\ \mathbf{1}-e_j &\in X & \text{for each } j. \end{aligned}$$

THEOREM 3. *If* $X \subseteq \{0, 1\}^m$ *is either a* W_0*-set or a* W_1*-set then* $F_C=F_P$.

Proof. For convenience, we adopt the canonical representations of W-sets in the preceding paragraph, and will let x, y, $\mathbf{0}$, and $\mathbf{1}$ denote column vectors in B^n. Given the hypothesis of the theorem, we assume that $f \in F_C$. Our first task is to prove that f_j is the same for all j. Letting $g=f_1=\cdots=f_m$, it is then shown that g is additive, so that f is additive. Linearity for f then follows from the argument used in the proof of Corollary 2, and $f \in F_P$ is implied by Theorem 2.

To verify $f_1=f_2=\cdots=f_m$, we need only show that $f_1=f_2$. Consider aggregation matrices

$$M_0=(y, \mathbf{1}-y, \mathbf{0},..., \mathbf{0}),$$

$$M_1=(y, \mathbf{1}-y, \mathbf{1},..., \mathbf{1}).$$

Since M_0 is in X^n for both W_0 and W_1, $f \in F_C$ implies for both cases that $f_1(y)=0 \Rightarrow f_2(1-y)=1$. Matrix M_0 for W_0 also gives $f_1(y)=1 \Rightarrow f_2(1-y)=0$, and matrix M_1 for W_1 yields the same implication. Since $X_j=\{0,1\}$ for both cases,

$$f_1(y)=1-f_2(1-y) \qquad \text{for all } y \in \{0,1\}^n.$$

This holds for every two indices. Therefore, since $m \geqslant 3$,

$$f_1(y)=1-f_3(1-y)=1-[1-f_2(y)]=f_2(y),$$

and hence $f_1=f_2$.

Let $g=f_1=\cdots=f_m$. We show next that

$$g(x)=g(y)=0 \Rightarrow g(x+y)=0, \tag{1}$$

so assume for now that x and y are fixed and that $g(x)=g(y)=0$. Let $N(\alpha\beta)=\{i$: the ith components of x and y are α and β, respectively$\}$ for $\alpha, \beta \in \{0,1\}$, and let 1_A be a column vector with 1 for all $i \in A$ and 0 otherwise.

To verify (1), assume first that $(1,1,0,\ldots,0) \notin X$, which must be true if X is the canonical W_0-set. Contrary to (1), suppose $g(x+y)=1$. Then the matrix

$$(1_{N(10)}, 1_{N(01)}, 1_{N(11)\cup N(00)}, \mathbf{0},\ldots,\mathbf{0})$$

implies either $g(1_{N(10)})=0$ or $g(1_{N(01)})=0$ since $g(\mathbf{0})=0$ and $g(1_{N(11)\cup N(00)})=g(1-(x+y))=0$. Assume for definiteness that $g(1_{N(10)})=0$. Then the matrix

$$(1_{N(10)}, 1_{N(01)\cup N(11)}(=y), 1_{N(00)}, \mathbf{0},\ldots,\mathbf{0})$$

implies $g(1_{N(00)})=1$ since its other aggregators are 0. Next,

$$(1_{N(00)}, 1_{N(01)}, 1_{N(10)\cup N(11)}(=x), \mathbf{0},\ldots,\mathbf{0})$$

gives $g(1_{N(01)})=0$, since otherwise its f is $(1,1,0,\ldots,0)$, which is not in X by prior assumption. Finally,

$$(1_{N(10)}, 1_{N(01)}, 1_{N(11)\cup N(00)}, \mathbf{0},\ldots,\mathbf{0})$$

yields a contradiction, since its f is $(0,0,\ldots,0)$.

This verifies (1) if $(1,1,0,\ldots,0) \notin X$, so assume now that $(1,1,0,\ldots,0) \in X$ and that X is a W_1-set. Then

$$(x, y, 1_{N(00)}, \mathbf{0},\ldots,\mathbf{0}) \in X^n,$$

and it implies $g(\mathbf{1}_{N(00)})=1$, since $g(x)=g(y)=g(\mathbf{0})=0$. In addition,

$$(\mathbf{1}_{N(00)}, \mathbf{1}, x+y, ..., x+y) \in X^n,$$

and to prevent its f from being (1, 1,..., 1), we require $g(x+y)=0$. This completes the proof of (1).

As explained in the opening paragraph of this proof, it remains only to show that g is additive. This follows from (1) if $g(x)=g(y)=1$, for we then have $g(\mathbf{1}-x)=g(\mathbf{1}-y)=0$, hence

$$0=g(\mathbf{1}-x+\mathbf{1}-y)=g(-x-y)=g(x+y).$$

Moreover, if $g(x)=0$ and $g(y)=1$, then $g(x+y)=1$, for otherwise $y=(x+y)+x$ and (1) imply $g(y)=0$. Since this covers all cases, the proof is complete. ▮

Remark 2. It is significant that the W_0 part of the preceding proof never uses an aggregation matrix with more than three nonconstant columns. This shows that, since the dictatorial functions for j that have $|X_j|=1$ are the same as all other consistent functions for those j, the F_P conclusion applies to all W_0-sets as originally defined. Moreover, it has another implication which bears on the question of the X's that imply $F_C=F_P$. In particular, we can use it to verify that there are X's that force $F_C=F_P$ and are neither W_0 nor W_1-sets. Consider $m=5$ and

$$X=\{e_j\}_1^5 \cup \{\mathbf{1}-e_j\}_1^5 \cup \{\mathbf{1}\},$$

which is not a W_1-set. The $\{e_j\}$ by themselves force $F_C=F_P$ and, since every $x \in X$ that is not an e_j has at least four 1's, the W_0 proof is unaffected by their presence. In other words, this X forces $F_C=F_P$ by the preceding W_0 proof.

6. Applications to Arrow Impossibility

We illustrate Theorem 3 by proving a variant of Arrow's theorem. Let $j=1, 2, 3$ correspond respectively to ba, ac, and cb. Let $x_{ij}=1$ if voter i prefers the first alternative in j to the second, with $x_{ij}=0$ for the opposite preference. The W_0-set situation is

	ba	ac	cb	preference order
e_1	1	0	0	bca
e_2	0	1	0	abc
e_3	0	0	1	cab

The three preference orders allowed by $\{e_1, e_2, e_3\}$ form a "cyclic" set. If voters' preferences are restricted to this set and the social order is also restricted to this set and satisfies C_1 (binary independence) and C_2 (Pareto optimality), then the W_0 part of Theorem 3 shows that one of the n voters dictates the social order.

To get the full flavor of Arrow's result, we form the W_1-set by adding the $\mathbf{1}-e_j$ to the e_j of the preceding paragraph. This generates the other three total orders on $\{a, b, c\}$, namely *bac*, *acb* and *cba*. The W_1 part of Theorem 3 then says that if $f \in F_C$, there is one i such that $f(x_1,..., x_n) = x_i$ for all $(x_1,..., x_n) \in [\{e_j\} \cup \{\mathbf{1}-e_j\}]^n$. Arrow's dictatorial conclusion for $m \geqslant 4$ is obtained by an immediate application of this result to overlapping three-candidate subsets that have two candidates in common.

7. Summary of $GF(2)$ and $m = 3$

Table III provides a summary of relationships between consistent aggregators and other types for all symmetric subsets $X \subseteq \{0, 1\}^3$ with three or more elements. We use two aggregators not defined previously, namely the *conjunctive aggregator*

$$F_\wedge = \{f \in F\text{: there is a nonempty } N \subseteq \{1,..., n\} \text{ such that, for all } j \leqslant m \text{ and all } (x_1,..., x_n) \in X^n, f_j(x_1,..., x_n) = \textstyle\prod_{i \in N} x_{ij}\},$$

TABLE III

Summary Implications for Consistent Aggregators for X in $\{0, 1\}^3$ and All $n \geqslant 2$

$\lvert X\rvert$	X	Type	Result	Reference
3	$\{100, 010, 001\}$	W_0	$F_C = F_P$	Theorem 3
	$\{110, 101, 011\}$	W_0	$F_C = F_P$	Theorem 3
4	$\{\mathbf{0}, 100, 010, 001\}$		$F_\wedge \subset F_C$	Compute
	$\{\mathbf{1}, 110, 101, 011\}$		$F_\vee \subset F_C$	Compute
	$\{\mathbf{0}, 110, 101, 011\}$	Hyperplane	$F_C = F_S$	Corollary 2
	$\{\mathbf{1}, 100, 010, 001\}$	Hyperplane	$F_C = F_S$	Corollary 2
5	$\{\mathbf{0}, \mathbf{1}, 100, 010, 001\}$		$F_C = F_\wedge$	[3]
	$\{\mathbf{0}, \mathbf{1}, 110, 101, 011\}$		$F_C = F_\vee$	[3]
6	$\{e_j\} \cup \{\mathbf{1}-e_j\}$	W_1	$F_C = F_P$	Theorem 3
7	$\{0, 1\}^3 \backslash \{\mathbf{0}\}$		$F_\vee \subset F_C$	Compute
	$\{0, 1\}^3 \backslash \{\mathbf{1}\}$		$F_\wedge \subset F_C$	Compute
8	$\{0, 1\}^3$		$F_S \subset F_C$	Lemma 1

and the *disjunctive aggregator*

$$F_\vee = \{f \in F\colon \text{there is a nonempty } N \subseteq \{1,\dots,n\} \text{ such that, for all } j \leqslant m \text{ and all } (x_1,\dots,x_n) \in X^n, f_j(x_1,\dots,x_n) = 1 - \textstyle\prod_{i\in N}(1 - x_{ij})\}.$$

The relevant citation to the text is given in the final column of the table, where "compute" invites the reader to check this result. The results for $|X| = 5$ apply to equivalence relations as discussed in Fishburn and Rubinstein [3].

The inclusions in the table are all proper. For example, the first X given for $|X| = 4$ has $f \in F_C$ when f is determined by majority quota $\alpha > \frac{1}{2}$, i.e., $f_j = 1$ if there are at least $n\alpha$ 1's in column j, but this f is not generally in $F_\wedge$. A majority quota $\beta > \frac{2}{3}$ gives the same result for the second X for $|X| = 7$. Note that the $F_C = F_P$ conclusion is valid *only* for Wilson sets.

References

1. K. J. Arrow, "Social Choice and Individual Values," Wiley, New York, 1951; 2nd ed., 1963.
2. R. F. Bordley, A multiplicative formula for aggregating probability assessments, *Manage. Sci.* **28** (1982), 1137-1148.
3. P. C. Fishburn and A. Rubinstein, Aggregation of equivalence relations, *J. Classification*, in press.
4. J. S. Kelly, "Arrow Impossibility Theorems," Academic Press, New York, 1978.
5. F. J. MacWilliams and N. J. A. Sloane, "The Theory of Error-Correcting Codes," North-Holland, Amsterdam, 1977.
6. K. O. May, Intransitivity, utility, and the aggregation of preference patterns, *Econometrica* **22** (1954), 1–13.
7. R. Wilson, On the theory of aggregation, *J. Econ. Theory* **10** (1975), 89–99.

Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium