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TIME PREFERENCE*

BY PETER C. FISHBURN AND ARIEL RUBINSTEIN[1]

1. INTRODUCTION

The aim of this paper is to examine the effect of the time of realization of an outcome on the relative desirability of the outcome. Three situations illustrate the comparisons involved in our study. In each situation, which alternative would you choose?

Situation 1: Get \$1000 today, or \$2000 one year from now;

Situation 2: Have a painless but badly decayed tooth pulled today, or wait a year and then have the original tooth plus its subsequently affected neighbor extracted together;

Situation 3: Get \$100 today, or flip a fair coin today and get \$50 tomorrow if the coin lands "heads" or \$150 one month from now if the coin lands "tails."

These situations ask you to compare two or more time-dependent outcomes (x, t), where x is "get \$1000," "have a tooth pulled," and so forth, and t is the time at which x is obtained. The outcomes in situations 1 and 2 have opposite polarities: the outcomes in 1 are desirable, while those in 2 are presumably painful. Situation 3 adds a risk dimension. In 3, the time of the gamble's resolution is specified along with the potential payoff times.

Similar examples abound in more economically oriented settings, as in the revenue x realized by selling a capital good at time t, or the timing and amount of a lump-sum payment arrived at through a bargaining process or court proceeding. While such situations are often subject to uncertainties, we shall suppress this factor to concentrate on the "pure" theory of time preference.

Our purpose is to examine implications of various axioms for a preference-or-indifference relation $\succsim$ on $X \times T$ when X is viewed as a set of outcomes and T is a set of times at which an outcome can occur. For simplicity, X is taken as a nondegenerate real interval, the individual's preference is presumed to increase in x at any specified $t \in T$, and, when $0 \in X$, 0 is interpreted as the status quo (no loss, no gain) outcome. Likewise, T will be either a discrete set of successive nonnegative integers or an interval of nonnegative numbers, with $0, 1 \in T$ in all cases and with $t=0$ denoting the present, or "now." In the usual fashion, $(x, t) \succ (y, s)$ means that $(x, t) \succsim (y, s)$ and not $[(y, s) \succsim (x, t)]$, and $(x, t) \sim (y, s)$ means that $(x, t) \succsim (y, s)$ and $(y, s) \succsim (x, t)$. We shall be concerned with the effect of t on the relative desirability of x. Roughly speaking, this effect is embodied in the indifference ($\sim$)

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[1] On leave from the Department of Economics of the Hebrew University at the time this work was completed (1980–81).

curves in $X \times T$ when time is continuous, and in sequences of indifferent pairs when time is discrete.

Recent interest in time preference is indebted to Koopmans's [1960] work on impatience (preference for getting desired outcomes sooner) in an infinite-period consumption streams context. Koopmans was motivated by earlier discussions of Böhm-Bawerk [1912] and Fisher [1930], and his initial contribution led to a series of studies (Burness [1973], [1976]; Diamond [1965]; Jamison [1969]; Koopmans, et al. [1964]) on impatience, eventual impatience, time perspective, stationarity, and related concepts in infinite-period and continuous-time formulations of consumption streams. Several writers have examined preference over time streams with von Neumann-Morgenstern utilities (Bell [1974]; Fishburn [1965]; Keeney and Raiffa [1976]; Meyer [1970], [1977]), while others have investigated effects of the times of resolutions of uncertainties on preferences in ongoing processes (Drèze and Modigliani [1972]; Kreps and Porteus [1978], [1979]; Spence and Zeckhauser [1972]).

Our study differs from these in its focus on the realization of a single outcome at a particular time. We can of course view our (x, t) as a stream with outcome 0 at each time $t' \neq t$, and when this is done many of our axioms can be viewed as specializations of conditions used by Koopmans and others. At the same time, some of their axioms, such as Koopmans's [1960] Postulates 3 and 3′, have no force in our context because of its highly restricted domain. Lacking the powerful domain structure of the aforementioned studies, we require a different approach in much of our work and consequently arrive at somewhat different theorems, although the tenor of our results follows previous patterns.

There are, however, several precedents in the literature for the present approach. Lancaster [1963] applies ten postulates to $\succsim$ on $X \times T$ with X a set of multidimensional commodity bundles and $T = [0, \infty)$. His postulates involve order, monotonicity, impatience and stationarity conditions, among others. An additional axiom, which says that $(x, t) \sim (y, t)$ iff $(mx, t) \sim (my, t)$ for all $m > 0$, and which is of interest only when elements in X are multidimensional, is then introduced and alleged to imply (on the question of sufficiency, see Nachman [1975, footnote 4]) that indifference in $X \times T$ satisfies $(x, t) \sim (e^{-kt}x, 0)$ for fixed $0 < k < 1$. Hence the receipt of bundle x at time t is indifferent to receipt of the discounted bundle $e^{-kt}x$ today. Lancaster refers to k as the consumer's rate of time preference.

More recently, Nachman [1975] has extended the notion of risk aversion of Pratt [1964] and Arrow [1965] to the concept of temporal risk aversion in a formulation where x denotes wealth and $X \times T = R \times [0, \infty)$. Nachman assumes that gambles on $X \times T$ are ordered in preference by their expected utilities based on U on $X \times T$. Roughly speaking, U is temporally risk averse if the certainty equivalent at time s of a gamble at time $t > s$ is no greater than the gamble's expected wealth. The flavor of his analysis is conveyed by part of his first theorem: U is temporally risk averse iff U is risk averse at each t, and impatient iff U is concave in x at each t and decreases in t for each x.

Prakash [1977] also considers von Neumann-Morgenstern utilities on gambles defined on $X \times T = R \times [0, \infty)$. By assuming that for every $x \in X$ and $s, t \in T$ there exists a unique $y \in X$ at which $(y, s) \sim (x, t)$, Prakash notes that the individual's preference order on gambles is completely determined from his preference on $X \times T$ and his preferences on the subset of gambles at a fixed instant of time.

The present paper adopts a measurement-theoretic approach to time preference by considering axioms for $\gtrsim$ on $X \times T$ that are sufficient for numerical representations of $\gtrsim$ that are of interest in the time context. An important aspect of the analysis is the difference between the discrete-time and continuous-time formulations. The question of whether X contains a "null" outcome 0, where there is no time preference in the sense that $(0, t) \sim (0, s)$ for all $s, t \in T$, also plays a role in later developments.

The next section considers axioms that are sufficient for the existence of continuous u on $X \times T$ that represents $\gtrsim$. The axioms used there (order, monotonicity, impatience, continuity) will be adopted throughout most of the paper. Section 3 then shows that an additional stationarity axiom allows u to be written as $u(xt) = \alpha^t f(x)$ with $0 < \alpha < 1$ whether time is discrete or continuous. An interesting aspect of the stationarity representation is that, for fixed $\gtrsim$, α can be taken as any number between 0 and 1: our focus on singular outcomes rather than streams precludes unique determination of α. Given α, f is fairly rigidly determined in the continuous-time case but lacks nice uniqueness properties when time is discrete.

Since stationarity is such a strong condition, the fourth section examines representations that are separable in x and t but not as specific as $\alpha^t f(x)$. When $X = [0, 1]$, for example, we shall consider the separable form $u(x, t) = \rho(t) f(x)$ with $\rho > 0$ and ρ strictly decreasing (due to impatience). It is noted that the separable representation follows readily from a standard axiom in measurement theory (the Thomsen [1927] condition) when time is continuous, but not otherwise.

The difficulty with separability in the discrete case leads to a formulation with von Neumann-Morgenstern utility in Section 5. Within this formulation, U has a separable form if the preference order on gambles at a fixed time is independent of the time index. This is true whether time is continuous or discrete.

Since concave utility functions are often relevant to economic analysis, we discuss concavity briefly in the Section 6. We shall focus on concavity of f in the stationarity context of Section 3 since we have nothing to add to previous results for gambles.

The final section returns to Koopmans's [1960] finding that impatience can follow from other axioms. We shall prove there that a form of impatience follows from prior monotonicity and stationarity axioms in conjunction with a strong order-continuity assumption.

2. ORDER-PRESERVING UTILITY

The outcome-time structure used in our study is specified by the following axiom.

A0. *X is a nondegenerate real interval; T is either a set of successive nonnegative integers or an interval of nonnegative numbers, and* $0, 1 \in T$.

This section considers implications of four axioms for $\succsim$ on $X \times T$. The axioms apply to all $x, y \in X$ and all $s, t \in T$.

A1. *$\succsim$ is a weak order on $X \times T$;*

A2. *If $x > y$ then $(x, t) \succ (y, t)$;*

A3. *If $s < t$ then $x > 0 \rightarrow (x, s) \succ (x, t)$, $x = 0 \rightarrow (x, s) \sim (x, t)$, and $x < 0 \rightarrow (x, t) \succ (x, s)$;*

A4. *$\{(x, t) : (x, t) \succsim (y, s)\}$ and $\{(x, t) : (y, s) \succsim (x, t)\}$ are closed in the product topology on $X \times T$.*

Axioms A1, A2 and A4 are typical ordering, monotonicity and continuity axioms. The factor topologies in A4 are the relative usual topology when X or T is an interval, or the discrete topology when T is $\{0, 1, \ldots, n\}$ or $\{0, 1, 2, \ldots\}$.

The third axiom is a composite impatience/procrastination condition. If $0 \in X$, then A3 says that 0 is a time-neutral outcome: the individual is indifferent between getting 0 sooner or later. Given A2, we view positive outcomes as "desirable" and negative outcomes as "undesirable." The rest of A3 then says that realization of a desirable outcome is preferred sooner to later, and that realization of an undesirable outcome is preferred later to sooner. The $x > 0$ part of A3 is an impatience assumption; the $x < 0$ part is a procrastination assumption.

Our characterization of impatience for desirable outcomes holds to the spirit of impatience in Böhm-Bawerk [1912], Fisher [1930] and Koopmans [1960]. Recently, Olson and Bailey [1981, p. 1] have argued that, excepting a special case for infinite time horizons, the "case for positive time preference is absolutely compelling...both in the positive and normative senses" despite prior objections by Stigler and Becker [1977].

Psychologists also have an interest in time preference. For example, Yates and Watts [1975, p. 304] report an experiment in which money could be lost at different future times that "offers direct support for the position that when [money] outcomes are really perceived as aversive, in a substantial number of instances people will prefer deferred outcomes to more immediate ones." Their finding supports the procrastination part of A3. However, people who derive anticipatory pleasure by deferring valued outcomes or who limit anxiety by advancing aversive outcomes will violate A3.

Although A3 is open to exception within the context of the other axioms, it often seems reasonable and, we believe, deserves analytical examination.

THEOREM 1. *If* A0–A4 *hold then there is a real valued function u on $X \times T$ such that:*
(i) *for all $(x, t), (y, s) \in X \times T$, $(x, t) \succsim (y, s)$ iff $u(x, t) \geq u(y, s)$;*

(ii) *u is continuous and increasing in x, u is continuous in t if T is an interval, and u* $(x, \cdot)$ *decreases (is constant, increases) in t if* $x>0$ $(x=0,\ x<0)$.

PROOF. If T is an interval, the proof follows immediately from Proposition 4 in Debreu [1964] or Lemma 5.1 in Fishburn [1970]. Assume henceforth that T is discrete.

Suppose first that $0 \in X$. Let $v(x)$ be any continuous, increasing and bounded real valued function on X, and set $u(x, 0)=v(x)$. If $n \in T/\{0\}$ and $x>0$, A2 and A3 imply $(x, 0)>(x, n)>(0, n)\sim(0, 0)$, so A1 gives $(x, 0)>(x, n)>(0, 0)$. Then A4 implies that there is a unique $x' \in (0, x)$ such that $(x, n)\sim(x', 0)$. Similarly, if $x<0$, we get $x' \in (x, 0)$ with $(x, n)\sim(x', 0)$. Finally, let $x'=0$ if $x=0$ since $(0, n)\sim(0, 0)$ by A3, and in general set $u(x, n)=u(x', 0)=v(x')$. By A1 and the construction, part (i) of the theorem holds, and it is easily seen that part (ii) holds also.

Suppose next that $0 \notin X$, and for definiteness assume that $x>0$ for all $x \in X$. Let $v(x)$ be as described in the preceding paragraph, and set $u(x, 0)=v(x)$. Construct $u(x, 1)$ as follows. If $(y, 0)\sim(x, 1)$ for no $y, x \in X$ then our axioms require $(y, 0)>(x, 1)$ for all $y, x \in X$, and in this case let $u(x, 1)$ be a continuous, increasing and bounded function on X such that $u(y, 0)>u(x, 1)$ for all $y, x \in X$. If $(y, 0)\sim(x, 1)$ for some $y, x \in X$, then an upper segment of the X interval at $t=1$ will have corresponding $x' \in X$ over a lower segment of the X interval at $t=0$ where $(x, 1)\sim(x', 0)$, and for these pairs we take $u(x, 1)=u(x', 0)$. Let $X^*=\{x \in X: (y, 0)>(x, 1) \text{ for all } y \in X\}$. If $X^*=\emptyset$, we go to $t=2$. Otherwise, define $u(x, 1)$ for all $x \in X^*$ as continuous, increasing and bounded with $\sup\{u(x, 1): x \in X^*\} = \inf\{u(x, 1): x \in X \backslash X^*\}$ sor that $u(x, 1)$ is continuous over X. It is easily checked that if the lower end of X is closed, then the upper end of X^* is open, and if the lower end of X is open then the upper end of X^* is closed, and therefore $u(x, 1)$ will be strictly increasing over X.

If $2 \in T$, define $u(x, 2)$ on the basis of $u(x, 1)$ in the same way that $u(x, 1)$ was based on $u(x, 0)$, and repeat the procedure for $u(x, n+1)$ on the basis of $u(x, n)$ so long as $n+1 \in T$. It is easily seen that u as thus constructed satisfies the conclusions of the theorem. Q. E. D.

3. STATIONARITY

Our stationarity axiom is stated in the indifference mode. It applies to all $x, y \in X$ and all $t, s, t+\tau, s+\tau \in T$.

A5. *If* $(x, t) \sim (y, t+\tau)$ *then* $(x, s) \sim (y, s+\tau)$.

This asserts that indifference between two time-dependent outcomes depends only on the difference (τ) between the times. If the two times are advanced or deferred by the same amount, then indifference will be preserved. Some notion of stationarity underlies evaluations that are based on constant discount rates. However, we know of no persuasive argument for stationarity as a psychologically viable assumption. and will therefore consider other axioms in the next two sections.

Stationarity is clearly independent of impatience. For example, A5 is compatible with either A3 or its converse: aspects related to this are discussed in Fishburn [1970, Chapter 7]. The effect of A5 coupled with A3 and our other axioms is noted in the next theorem.

THEOREM 2. *If* A0–A5 *hold, then, given any* $0<\alpha<1$, *there is a continuous, increasing real valued function* f *on* X *such that*:

(i) *for all* $(x, t), (y, s) \in X \times T$, $(x, t) \gtrsim (y, s)$ *iff* $\alpha^t f(x) \geq \alpha^s f(y)$;

(ii) $f(0)$ *must be* 0 *if* $0 \in X$, *and* $xf(x)$ *must be positive for all* $x \in X \setminus \{0\}$;

(iii) *if* T *is an interval then* f *is unique (given* α*) up to multiplication by positive constants on* $\{x \in X : x>0\}$ *and on* $\{x \in X : x<0\}$.

If the given α is changed in the representation, then f also must be changed. For example, if $X=T=[0, 1]$ and f_α is the unique f—by (iii)—that satisfies the representation when $f_\alpha(1)=1$, then f_α and f_β are related as $f_\beta(x)=[f_\alpha(x)]^k$ with $k=\log \beta/\log \alpha$. This follows easily from the proof of Theorem 2 for the continuous-time case.

It may also be noted that if all outcomes in X are positive, then $\alpha^t f(x)$ can be put in additive form by taking logarithms: for example, we get $(x, t) \gtrsim (y, s)$ iff $g(x)-t \geq g(y)-s$.

PROOF OF THEOREM 2. Let A0–A5 hold, fix α in (0, 1), and assume with no loss in generality that $x>0$ for some $x \in X$. If X has negative outcomes, these can be dealt with independently of nonnegative outcomes (by A3 and A2) in a proof that mimics what follows. Henceforth in this proof we assume that $x \geq 0$ for all $x \in X$.

The proof for $\alpha^t f(x)$ will be split between the discrete-time and continuous-time cases. A number of details that the reader can easily supply will be omitted to keep matters relatively brief.

Discrete-time proof. Let $a=\inf X \geq 0$ and $b=\sup X$. If $(X, 0)>(X, 1)$, i.e., $(x, 0)>(y, 1)$ for all $x, y \in X$, then A1–A4 require $0 \notin X$, and A5 implies $(X, n)>(X, n+1)$ whenever $n+1 \in T$. In this case let f be any continuous and increasing function on X for which $\inf f(X)=\beta$ and $\sup f(X)=1$ with $\alpha<\beta<1$. Then $\alpha^n f(x)>\alpha^{n+1} f(y)$ for all $x, y \in X$, so (i) of Theorem 2 holds. Moreover, (i) requires $f>0$, for otherwise we get $f(x) \leq \alpha f(x)$ for some x, hence $(x, 1) \gtrsim (x, 0)$ by (i), which by A3 implies $x \leq 0$.

Henceforth in the discrete proof, suppose there is some indifference between $t=0$ and $t=1$, and for definiteness take $(x_1, 0) \sim (1, 1)$ with $0<x_1$ and $a \leq x_1<1 \leq b$ by A1–A4. Using (1, 1) and (1, 0), construct a dual sequence $\ldots x_2, x_1, 1, x_1', x_2', \ldots$ of decreasing x_i and increasing x_i' that satisfies

$$(1, 1) \sim (x_1, 0), (x_1, 1) \sim (x_2, 0), \ldots;$$

$$(1, 0) \sim (x_1', 1), (x_1', 0) \sim (x_2', 1), \ldots$$

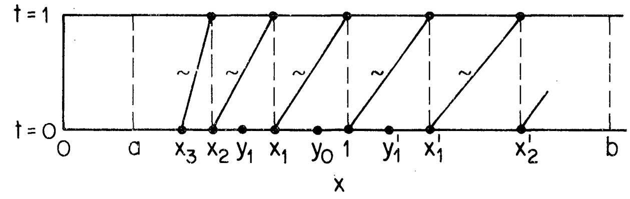

FIGURE 1

as illustrated in Figure 1. The x_i part of the dual sequence is nonempty: its behavior depends in part on the lower end of X. We enumerate possibilities.

L1. $a=0$ with $0 \in X$. Then $x_1, x_2, \ldots$ is denumerable and $x_m \downarrow 0$ (x_m goes to 0 from above as $m \to \infty$). See the second paragraph of the proof of Theorem 1, and note that $x_m \downarrow c > 0$ yields a contradiction.

L2. $a > 0$ and $a \in X$. Then $x_1, x_2, \ldots$ must be finite.

L3. $a \notin X$. Then $x_1, x_2, \ldots$ can be either finite or denumerable. If denumerable then $x_m \downarrow a$.

The x_i' part of the dual sequence could be empty, nonempty and finite, or denumerable. If $b \in X$ then $x_1', x_2', \ldots$ must be finite, and if x_m' is the last term then $(x_m, 0) > (b, 1)$. If $x_1', x_2', \ldots$ is denumerable then $x_m' \to b$ as $m \to \infty$.

Let y_0 be any value in $(x_1, 1)$ as shown in Figure 1, and construct a dual sequence $\ldots, y_2, y_1, y_0, y_1', y_2', \ldots$ in the manner described earlier: $(y_0, 1) \sim (y_1, 0), (y_1, 1) \sim (y_2, 0), \ldots; (y_0, 0) \sim (y_1', 1), (y_1', 0) \sim (y_2', 1), \ldots$. By A1–A4, the two dual sequences mesh as

$$\cdots < y_2 < x_2 < y_1 < x_1 < y_0 < 1 < y_1' < x_1' < \cdots$$

Let S denote the family of all dual sequences constructed in this way on the basis of y_0 for all $y_0 \in (x_1, 1]$. Every two distinct sequences in S intermesh in the indicated manner, and every positive $x \in X$ appears in exactly one sequence in S.

Define f on X as follows. If $0 \in X$, set $f(0)=0$, as required for (i) by A3, and set $f(1)=1$. Then for the sequence $\ldots, x_2, x_1, 1, x_1', x_2', \ldots$ let

$$f(x_i) = \alpha^i$$
$$f(x_j') = \alpha^{-j}$$

for all applicable i and j, and note that, given α and $f(1)=1$, these are the only possible values that will satisfy (i). Next, define f on $(x_1, 1)$ as any continuous and increasing function with $\inf f((x, 1)) = \alpha = f(x_1)$ and $\sup f((x, 1)) = 1 = f(1)$. Finally, extend f for each dual y_0 sequence in S for $x_1 < y_0 < 1$ in the only way possible to satisfy (i):

$$f(y_i) = \alpha^i f(y_0)$$
$$f(y_j') = \alpha^{-j} f(y_0) .$$

The intermeshing and covering aspects of S ensure that f is positive, increasing and continuous for $x>0$. If $0 \in X$, L1 shows that f is continuous at 0. Moreover, all $x>0$ must have $f(x)>0$ if (i) is to hold.

It follows readily from the construction of f that part (i) of Theorem 2 holds for all $(x, t), (y, s) \in X \times \{0, 1\}$. By A5, the structure of $\gtrsim$ on $(X, 2)$ versus $(X, 1)$ is precisely the same as the structure of $\gtrsim$ on $(X, 1)$ versus $(X, 0)$ when $2 \in T$. With $u(x, 1) = \alpha u(x, 0) = \alpha f(x)$, this means that $u(x, 2) = \alpha u(x, 1) = \alpha^2 f(x)$ satisfies part (i) of the theorem for $t=2$ versus $t=1$. It then follows from transitivity that $u(x, t) = \alpha^t f(x)$ preserves $\gtrsim$ on $X \times \{0, 1, 2\}$. The same argument applied to successive integers in T shows that (i) holds on $X \times T$.

Continuous-time proof. For convenience, assume that $1 \in X$. Let $[a, b]$ be a bounded closed interval within X with $1 \in [a, b]$, and let $[0, t^*]$ be a bounded interval in T, with $a<b$ and $0<t^*$. Set $f(1)=1$. We shall note that f is then uniquely determined on $[a, b]$ in such a way that $\alpha^t f(x)$ represents $\gtrsim$ on $[a, b] \times [0, t^*]$. We can then expand $[a, b]$ and $[0, t^*]$ as necessary in a countable number of steps so that every point in $X \times T$ is covered by some $[a, b] \times [0, t^*]$. An f with $f(1)=1$ is defined to satisfy (i) at each step: by uniqueness, each new f is identical to its predecessors on their common domain. We thus get f defined on all of $X \times T$ to satisfy (i), and it is unique, given $f(1)=1$. If the value of $f(1)$ is changed, say to $\lambda>0$, then our construction shows that all other f values must be multiplied by λ to preserve (i), so f is unique up to multiplication by a positive constant. (A similar result holds for f defined on negative x when X goes below 0. Different positive constants can be used in the two domains.) Clearly, f must be increasing when (i) holds, and it follows from the construction that f is continuous.

We work henceforth in $[a, b] \times [0, t^*]$. This region is covered by continuous indifference curves of the type shown in Figure 2. The curve through $(b, 0)$ is a

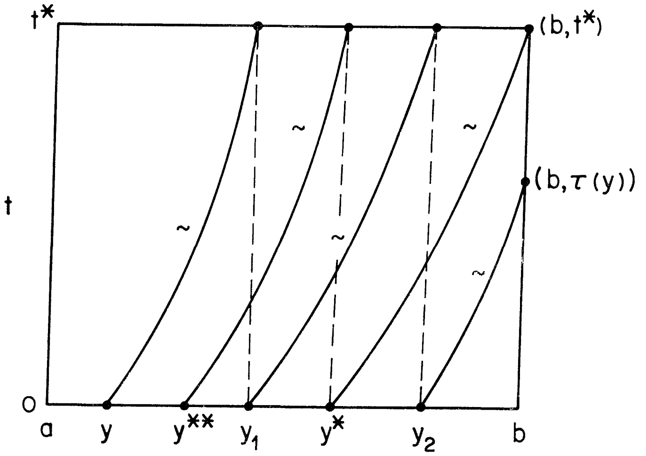

FIGURE 2

single point at $(b, 0)$, and the same is true at (a, t^*) unless $a=0$, in which case the vertical line between $(0, 0)$ and $(0, t^*)$ is an indifference curve by A3, and any sequence like y^*, y^{**},... will be denumerable and approach 0 (cf. L1 in discrete proof). If $a>0$, then the boundary point that is indifferent to (b, t^*) at the upper right corner can either be $(y^*, 0)$, as shown on the figure, or a point (a, t) for $0 \leq t < t^*$. When $a>0$, a sequence like y^*, y^{**},... must be finite (cf. L2).

Let $\tau(b)=0$, and for each $y \in [a, b)$ with $y>0$ let $\tau(y)$ be the unique t in $(0, t^*]$ such that either $(y, 0) \sim (b, \tau(y))$ or else there is a finite sequence $y_1, y_2, \ldots, y_n$ such that $(y, 0) \sim (y_1, t^*)$, $(y_1, 0) \sim (y_2, t^*), \ldots, (y_{n-1}, 0) \sim (y_n, t^*)$ [with $y_n < b$] and $(y_n, 0) \sim (b, \tau(y))$. Figure 2 illustrates the latter for $n=2$. Let

$$G(y) = (0, \tau(y)) \qquad \text{if} \quad (y, 0) \sim (b, \tau(y))$$

$$G(y) = (n, \tau(y)) \qquad \text{if} \quad (y, 0) \sim (y_1, t^*), \ldots, (y_n, 0) \sim (b, \tau(y))$$

for all positive y in $[a, b]$. If $G(y)=(n, \tau(y))$ for $y>0$, then part (i) of Theorem 2 requires

$$f(y) = (\alpha^{t^*})^n \alpha^{\tau(y)} f(b) = \alpha^{nt^* + \tau(y)} f(b) \, .$$

Given $f(1)=1$, this equation determines $f(b)$ and then determines $f(y)$ for every positive y in $[a, b]$. If $a=0$, then $f(0)=0$ as before. We see that f is continuous and increasing on $[a, b]$. (Continuity needs to account for the change from n to $n+1$ in $G(y)=(n, \tau(y))$, but there are no unusual problems with this.)

Our proof for continuous time is essentially complete if $\alpha^t f(x)$ as just defined is constant on each indifference curve in $[a, b] \times [0, t^*]$, for then (i) will indeed hold on the rectangle. In comparing (x, t) and (y, s), (i) clearly holds if either x or y is 0, so assume that both x and y are positive. Given $(x, t) \neq (y, s)$, we can have $(x, t) \sim (y, s)$ only if $(y<x, s<t)$ or $(x<y, t<s)$, so assume for definiteness that $(y<x, s<t)$. Let z_1 be the value of x on the lower or left boundary of the rectangle that gives indifference with (b, t^*). That is, $z_1 \in \{y^*, a\}$ as shown in Figure 2. With $y<x$, suppose first that $z_1 \leq y$. Then

(A5) $\quad (x, t) \sim (y, s) \text{ iff } (x, t-s) \sim (y, 0)$

(A1) $\quad \text{iff } (x, t-s) \sim (b, \tau(y))$

(A5) $\quad \text{iff } (x, 0) \sim (b, \tau(y) - t + s)$

(A1) $\quad \text{iff } (b, \tau(x)) \sim (b, \tau(y) - t + s)$

(A2) $\quad \text{iff } \tau(x) = \tau(y) - t + s$

$\quad \text{iff } \alpha^t \alpha^{\tau(x)} = \alpha^s \alpha^{\tau(y)}$

$\quad \text{iff } \alpha^t f(x) = \alpha^s f(y) \quad (\text{def. of } f)$.

If $z_1=a$, this completes the proof. Suppose henceforth that $a<z_1=y^*$. Let z_2 be the x value at the lower or left boundary of the point indifferent to (y^*, t^*), so that $z_2 \in \{a, y^{**}\}$. We suppose next that $z_2 \leq y < z_1$. If $z_1 \leq x$ then, with $G(y)=(1, \tau(y))$ and $(y, 0) \sim (y_1, t^*)$, (A5) $(x, t) \sim (y, s)$ iff $(x, t - s) \sim (y, 0)$

(A1) iff $(x, t - s) \sim (y_1, t^*)$

iff $\tau(x) + t - s = t^* + \tau(y)$ (see above)

iff $\alpha^t f(x) = \alpha^s f(y)$ (def. of f);

and if $z_2 \leq y < x < z_1$ with $G(x) = (1, \tau(x))$ and $(x, 0) \sim (x_1, t^*)$, then

(A5) $(x, t) \sim (y, s)$ iff $(x, t - s) \sim (y, 0)$

(A1) iff $(x, t - s) \sim (y_1, t^*)$

(A5) iff $(x, 0) \sim (y_1, t^* - t + s)$

(A1) iff $(x_1, t^*) \sim (y_1, t^* - t + s)$

iff $\tau(x) + t^* = \tau(y) + t^* - t + s$ (first case)

iff $\alpha^t f(x) = \alpha^s f(y)$ (def. of f).

If $z_2 = a$, this completes the proof. Otherwise, we continue in the manner just indicated (define z_3, use results just proved,...) until all possible $x, y > 0$ in $[a, b]$ have been covered. Q. E. D.

4. SEPARABILITY

Although stationarity may fail to hold in many cases that seem suitable for A0–A4, weaker conditions can lead to representations for $\gtrsim$ which separate the effect of time preference from outcome preference. We shall comment here on one such condition, which has been used previously by Debreu [1960] and others for additive measurement representations. The following Thomsen condition applies to all $x, y, z \in X$ and all $r, s, t \in T$.

A6. *If* $(x, t) \sim (y, s)$ *and* $(y, r) \sim (z, t)$ *then* $(x, r) \sim (z, s)$.

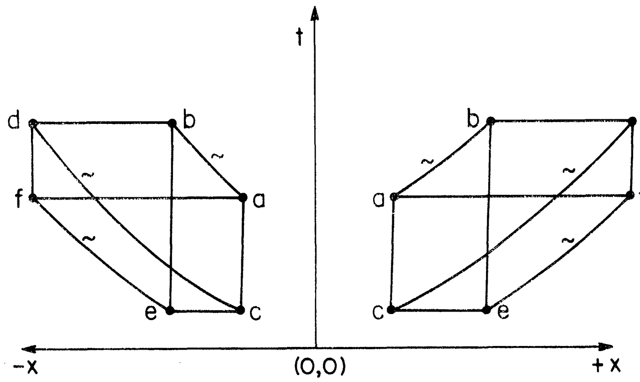

FIGURE 3

Because A3 and other axioms prohibit $(x, t) \sim (y, s)$ when $x \leq 0 < y$ or $x < 0 \leq y$, interesting occasions of A6 arise only when all of x, y and z have the same sign. Figure 3 illustrates the axiom for positive and negative outcomes. In either domain, A6 says that $\{a \sim b, c \sim d\} \rightarrow e \sim f$, and $\{a \sim b, e \sim f\} \rightarrow c \sim d$.

Although A6 is considerably weaker than A5 in the context of A0–A4, there seems to be no simple defense of A6 in this context. However, the next section offers justification for A6 within a richer structure of preference: we shall come to that soon. Meanwhile, we remark that A6 is necessary for the type of separability considered here and will now prove that it is sufficient when time is continuous.

The following theorem is stated only for nonnegative outcomes, for expository convenience. A similar representation holds for negative outcomes. If there are both positive and negative outcomes, we get say $u(x, t) = \rho(t)f(x)$ for $x \geq 0$ and $u(x, t) = \sigma(t)f(x)$ for $x \leq 0$, where f is continuous and increasing with $f(0) = 0$, and each of ρ and σ are positive, continuous and decreasing. Because $x = 0$ in A3 separates the positive and negative regions, there need be no special relationship between ρ and σ.

THEOREM 3. *Suppose* A0–A4 *and* A6 *hold, T is an interval, and* $X \geq 0$. *Then there are continuous real valued functions f on X and ρ on T such that:*

(i) *for all* $(x, t), (y, s) \in X \times T$, $(x, t) \gtrsim (y, s)$ *iff* $\rho(t)f(x) \geq \rho(s)f(y)$;

(ii) *f is increasing with* $f(0) = 0$ *if* $0 \in X$, *and ρ is decreasing and positive*;

(iii) *f' on X and ρ' on T satisfy* (i) *and* (ii) *along with f and ρ iff there are positive numbers k, k_1 and k_2 such that* $f' = k_1 f^k$ *and* $\rho' = k_2 \rho^k$.

PROOF. Given the hypothesis of the theorem, let $X^+ = \{x \in X : x > 0\}$. It then follows readily from Theorem 5.4 in Fishburn [1970] that there are continuous real valued functions F on X^+ and G on T such that, for all (x, t), $(y, s) \in X^+ \times T$

$$(x, t) \gtrsim (y, s) \qquad \text{iff} \quad F(x) + G(t) \geq F(y) + G(s),$$

with F and G unique up to similar positive affine transformations. That is, F' and G' represent $\gtrsim$ along with F and G if and only if there are numbers $k > 0$, b_1 and b_2 such $F' = kF + b_1$ and $G' = kG + b_2$. By A2 and A3, F must increase in x, and G must decrease in t. In addition, if $0 \in X$, then $F(x)$ must approach $-\infty$ as $x \downarrow 0$: cf. L1 in the discrete proof of Theorem 2.

Let c be any number that exceeds 1, and define

$$f(x) = c^{F(x)} \qquad \text{for all} \quad x \in X^+$$
$$\rho(t) = c^{G(t)} \qquad \text{for all} \quad t \in T,$$

and (for A3) set $f(0) = 0$ if $0 \in X$. Then (i) and (ii) of the theorem follow.

For (iii), suppose $\{f, \rho\}$ and $\{f', \rho'\}$ satisfy (i) and (ii). Given any $c > 1$, let $F = \log_c f$ and $F' = \log_c f'$ on X^+, and let $G = \log_c \rho$ and $G' = \log_c \rho'$ on T. By the uniqueness result in the preceding paragraph there are $k > 0$, b_1 and b_2 such that

$F' = kF + b_1$ and $G' = kG + b_2$. Therefore $f' = c^{b_1} f^k$ and $\rho' = c^{b_2} \rho^k$, which gives part of (iii) with $k_i = c^{b_i}$. Moreover, if f and ρ satisfy (i) and (ii), and if $f' = k_1 f^k$ and $\rho' = k_2 \rho^k$ with k, k_1 and k_2 positive, then f' and ρ' also satisfy (i) and (ii).

Q. E. D.

The separable form $\rho(t)f(x)$ to represent $\succsim$ can be axiomatized in the discrete-time setting (Jaffray [1974]; Tversky [1967]) without invoking a richer formulation, but the necessary axioms are too complex to interest us here. Related work by Fishburn [1981] suggests that there is no set of simple necessary axioms for separability within the discrete-time context of A0–A4.

5. UTILITY INDEPENDENCE WITH GAMBLES

To obtain a different perspective on separability, we now consider a gambles formulation after the fashion of Nachman [1975] and Prakash [1977]. Notationally, we shall let P and P_0 be respectively the sets of all gambles (simple probability distributions) on $X \times T$ and on X. For any $p \in P_0$ and $t \in T$, (p, t) denotes the gamble in P that yields the time-dependent outcome (x, t) with probability $p(x)$. To avoid possible ambiguities concerning the time at which the result of a chosen gamble in P is known, we shall presume that any chosen gamble is played out at time 0. Thus, if a gamble in P yields ($1000, 1 year from today) or ($2000, 2 years from today), it is to be understood that if this gamble is chosen (now) then its outcome will be determined, but not yet realized, today.

We shall assume that $\succsim$ on P satisfies axioms (Fishburn [1970]; Herstein and Milnor [1953]) that are necessary and sufficient for the existence of von Neumann-Morgenstern utilities. For convenience, we simply assume

B1. *There is a real valued function U on $X \times T$ such that expected utilities based on U of gambles in P preserve $\succsim$ on P.*

The restriction of $\succsim$ to degenerate (one-point) gambles will be assumed to satisfy A1–A4 under the usual convention that $(x, t) \succsim (y, s)$ means that the gamble in P with probability 1 for (x, t) is preferred or indifferent to the gamble in P with probability 1 for (y, s).

The main separability axiom that we shall use in the present formulation is a "utility independence" condition (Keeney, [1968]; Pollak, [1967]). It applies to all $p, q \in P_0$ and all $s, t \in T$.

B2. *If $(p, s) \succsim (q, s)$ then $(p, t) \succsim (q, t)$.*

This says that if p is as preferred as q from the perspective of the present when the resultant outcome is to be realized at time s, then p is as preferred as q from the perspective of the present when the resultant outcome is to be realized at time t. If you prefer the even-chance gamble between -$100 and $1,500 to the sure thing of $200 when the actual payoff occurs tomorrow, then B2 says that you will prefer the gamble to the sure thing when the actual payoff occurs one year from now.

Although anticipated future circumstances could vitiate B2, it seems to be a good approximation to reality in many situations. The effect of B2 on U is noted in the following lemma.

LEMMA 1. *Suppose* A0, B1 *and* B2 *hold with U as specified in* B1. *Then there are real valued functions f on X and ρ and ω on T with $\rho > 0$ such that, for all* $(x, t) \in X \times T$,

$$U(x, t) = \rho(t)f(x) + \omega(t).$$

PROOF. The result is well known (Keeney [1968]; Keeney and Raiffa [1976]; Pollak [1967]). Q. E. D.

We intimated in the preceding section that our new axioms imply the Thomsen condition A6. This is in fact true if $0 \in X$ and if A3 holds, for then we require $U(0, t) = U(0, 0)$ for all $t \in T$ so that, with $c = U(0, 0)$, Lemma 1 gives

$$U(x, t) = \rho(t)\,[f(x) - f(0)] + c,$$

and A6 is easily seen to hold for this U. However, A6 can fail when $0 \notin X$. For example, suppose $X = [1, \infty)$ and $\gtrsim$ on P is determined by the order of expected utilities based on

$$U(x, t) = 2^{-t}x - t/(t+1).$$

Then A0–A4, B1 and B2 hold, but A6 is false. A companion to B2 that is considered below shows however that A6 follows in general if we adopt another axiom.

THEOREM 4. *Suppose* A0, A2–A4, B1 *and* B2 *hold. Then there are real valued functions f on X and ρ and ω on T such that*:
(i) *expected values based on $\rho(t)f(x) + \omega(t)$ preserve $\gtrsim$ on P;*
(ii) *$\rho > 0$ and f is increasing with $f(0) = 0$ if $0 \in X$ and $xf(x) > 0$ whenever $x \neq 0$;*
(iii) *if $0 \in X$ then $\omega \equiv 0$ and ρ is decreasing, and if, in addition, T is an interval, then f and ρ are continuous.*

PROOF. The stated conclusions follow easily from Lemma 1, shifts in the origin of U (see the paragraph following the lemma), and A2–A4. Continuity in (iii) follows by analogy with Theorem 3 since we know that A6 holds when $0 \in X$. Q. E. D.

An axiom like B2 that reverses the roles of x and t might also be considered in the gambles context. If $x, y \in X$ and if ν and μ are simple probability distributions on T, then (taking account of A3), we could posit the following, where (x, ν) is a gamble that yields x at a time determined by the distribution ν.

B3. *If $xy > 0$ and $(x, \nu) \gtrsim (x, \mu)$ then $(y, \nu) \gtrsim (y, \mu)$.*

If B3 holds and you are indifferent between an even-chance gamble that yields \$100 either today or a year from now and \$100 as a sure thing three months from now, then you will be indifferent between an even-chance gamble that yields \$500 either today or a year from now and \$500 as a sure thing three months from now.

It is easily seen from Theorem 4 that its axioms imply B3 whenever $0 \in X$. If $0 \notin X$, say with $X > 0$, we know (Keeney and Raiffa [1976]) that the axioms of Lemma 1 in conjunction with B3 imply that $U(x, t)$ can be written either as the sum of functions of x and t, or as the product of functions of x and t. Hence, if $0 \notin X$ and if A0–A4 and B1–B3 hold, then A6 must hold.

Consequently, we see that A6 holds regardless of whether 0 is in X if one accepts B3 along with the axioms used in Theorem 4.

6. REMARKS ON CONCAVITY

This section comments briefly on the possibility of having f concave when the stationarity representation $u(x, t) = \alpha^t f(x)$ holds as specified in Theorem 2. Although there is a sense of arbitrariness in asking about concavity of f in this particular form, there are contexts, such as bargaining (Rubinstein, [1982]), where it is very natural. For simplicity, we assume that $X = [0, 1]$ with T discrete. Let δ and g be defined by

$$(x + \delta(x), 1) \sim (x, 0)$$
$$g(x) = x + \delta(x).$$

Given $(1, 1) \sim (x_1, 0)$ as in Figure 1, δ and g are defined on $[0, x_1]$ with $\delta(0) = g(0) = 0$, $\delta(x_1) = 1 - x_1$ and $g(x_1) = 1$. By A1–A4, g increases and is continuous on $[0, x_1]$, and $\delta(x) > 0$ for $0 < x \leq x_1$.

We consider two conditions on δ. In the second, $\delta'(x_1^-) = \lim \{[\delta(x_1) - \delta(x)]/(x_1 - x) : x \uparrow x_1\}$.

CONDITION 1. *δ is an increasing function;*

CONDITION 2. *δ is concave and $\delta'(x_1^-) > 0$.*

Condition 1 seems plausible, while Condition 2 is more demanding. The following proposition relates the possibility of concave f to the conditions.

PROPOSITION 1. *Suppose $\succsim$ on $[0, 1] \times \{0, 1, \ldots, n\}$ satisfies* A1–A5. *For each $0 < \alpha < 1$, let F_α be the set of all functions f_α on $[0, 1]$ such that $\alpha^t f_\alpha(x)$ represents $\succsim$ on $X \times T$ as specified in Theorem 2. Then Condition 1 is necessary but not sufficient, and Condition 2 is sufficient, for the existence of an $\alpha \in (0, 1)$ for which some $f_\alpha \in F_\alpha$ is concave.*

PROOF. Given the hypotheses of the proposition, we consider the three conclusions in turn.

Necessity of Condition 1. Assume that $f \in F_\alpha$ for some α is concave and take $x_1 \geq y > x > 0$. Since $f(z) = \alpha f(g(z))$,

$$\frac{f(y) - f(x)}{y - x} = \alpha \left[\frac{f(g(y)) - f(g(x))}{g(y) - g(x)} \right] \left[\frac{g(y) - g(x)}{y - x} \right].$$

Since f is concave, the left hand side of this is as great as the first ratio on its right hand side, and therefore $[g(y)-g(x)]/(y-x)>1$, or $\delta(y)>\delta(x)$.

Insufficiency of Condition 1. We shall define g so that Condition 1 holds but no f_α can be concave. Let $x_0=1$ and $x_i=2^{-i}$ with $g(x_i)=x_{i-1}$ for $i\geq 1$, and define y_i and $g(y_{i+1})$ for $i=0, 1,\ldots$ by

$$y_i = x_i - (x_i - x_{i+1})\lambda^{i+1}$$
$$g(y_{i+1}) = y_i$$

with $0<\lambda<1$. The definition of g is completed by $g(0)=0$ along with linear interpolation between x_1 and y_1, y_1 and $x_2,\ldots$. For $i\geq 1$,

$$\begin{aligned}\delta(x_i) &= g(x_i) - x_i = x_{i-1} - x_i = 2^{-i} = x_i,\\ \delta(y_i) &= y_{i-1} - y_i\\ &= x_{i-1} - (x_{i-1} - x_i)\lambda^i - x_i + (x_i - x_{i+1})\lambda^{i+1}\\ &= x_i - \lambda^i x_i + \lambda^{i+1}x_{i+1}\\ &= (1 - \lambda^i)x_i + \lambda^{i+1}x_{i+1}.\end{aligned}$$

Therefore, for $i\geq 1$, $\delta(x_{i+1})<\delta(y_i)<\delta(x_i)$ iff $x_{i+1}<(1-\lambda^i)x_i+\lambda^{i+1}x_{i+1}<x_i$ iff $\{1-\lambda^{i+1}<2(1-\lambda^i)$ and $\lambda^{i+1}<2\lambda^i\}$, which is true. Hence Condition 1 holds. Suppose f for α is concave. Then

$$\frac{f(y_i)-f(x_{i+1})}{y_i-x_{i+1}} \geq \frac{f(x_i)-f(y_i)}{x_i-y_i}$$

so that

$$\begin{aligned}\frac{\lambda^{i+1}}{1-\lambda^{i+1}} = \frac{x_i-y_i}{y_i-x_{i+1}} \geq \frac{f(x_i)-f(y_i)}{f(y_i)-f(x_{i+1})} &= \frac{\alpha^{-1}[f(x_{i-1})-f(y_{i-1})]}{\alpha^{-1}[f(y_{i-1})-f(x_i)]}\\ &\;\;\vdots\\ &= \frac{f(x_1)-f(y_1)}{f(y_1)-f(x_2)},\end{aligned}$$

which is impossible since $\lambda^{i+1}/(1-\lambda^{i+1})$ goes to 0 as i increases. Hence no f_α for any α can be concave.

Sufficiency of Condition 2. Given concave δ with $\delta'(x_1^-)>0$, let $\alpha=1/g'(x_1^-)$. With $x_0(=1)$, x_1, $x_2,\ldots$ as before (Figure 1), set $f(x_0)=1, f(x_1)=\alpha$, and define f on $[x_1, x_0]$ by linear interpolation. For any $0<x<x_1$, define f to satisfy $f(x)=\alpha f(g(x))$ with $(g(x), 1)\sim(x, 0)$. Concavity of f in $[x_1, x_0]$ and of g imply that f is concave in each $[x_{i+1}, x_i]$. Working with left and right derivatives, concavity of f holds everywhere if $f'(x_i^-)\geq f'(x_i^+)$ for $i\geq 1$. We have $f'(x_i^-)=\alpha f'(x_{i-1}^-)g'(x_i^-)$ and $f'(x_i^+)=\alpha f'(x_{i-1}^+)g'(x_i^+)$. At x_1, the definition of α and linearity on $[x_1, x_0]$ give

$$f'(x_1^-) = \alpha f'(x_0^-)g'(x_1^-) = f'(x_0^-) = f'(x_1^+) = 1.$$

Induction then gives $f'(x_i^-)=\alpha f'(x_{i-1}^-)g'(x_i^-)\geq\alpha f'(x_{i-1}^+)g'(x_i^+)=f'(x_i^+)$ for $i=2, 3, \ldots$. Q.E.D.

7. A DERIVATION OF IMPATIENCE

Towards the end of his pioneering study, Koopmans [1960, p. 306] says that "...impatience was introduced by Böhm-Bawerk as a psychological characteristic of human economic preference in decisions concerning (presumably) a *finite* time horizon. It now appears that impatience...is also a necessary logical consequence of more elementary properties of a utility function of programs with an *infinite* time horizon..." We shall conclude our own study by demonstrating that (weak) impatience for positive outcomes is implied by monotonicity, the 0 part of A3, stationarity, and the following order-continuity condition:

A7. *There is a real valued function u on $X\times T$ that preserves $\gtrsim$ and is such that for every $\varepsilon>0$ and every number u_0 there is a $\gamma>0$ such that, for all $x, y\in X$ and all $t\in T$ for which $u(x, t)=u_0$ and $|y-x|<\gamma$, it is true also that $|u(y, t)-u_0|<\varepsilon$.*

This adds a sense of uniform continuity to A1 and A4 — which are implied by A7 — and is similar to conditions used by Koopmans. It is a rather strong condition, but something like it is needed to induce impatience. As one might expect, a dual version of the following theorem yields a (weak) procrastination conclusion for negative outcomes.

THEOREM 5. *Suppose $X\times T=[0, 1]\times\{0, 1, 2,\ldots\}$, that* A2, A5 *and* A7 *hold, and that $(0, s)\sim(0, t)$ for all $s, t\in T$. Then, for all $x>0$ and all $s, t\in T$, $s<t$ implies $(x, s)\gtrsim(x, t)$.*

PROOF. Let the hypotheses hold, but suppose the conclusion is false so that $(x, n_1)>(x, n_2)$ for some $x>0$ and $n_1, n_2\in T$ with $n_1>n_2$. Since our axioms and $(x, 0)\gtrsim(x, 1)$ imply $(x, n_2)\gtrsim(x, n_1)$, we require $(x, 1)>(x, 0)$. Since $(x, 1)>(x, 0)>(0, 0)\sim(0, 1)$ by A2 and the 0 part of A3, $(x, 1)>(x, 0)>(0, 1)$, so, by A7, there is a y strictly between 0 and x such that $(y, 1)\sim(x, 0)$. Let $x_0=x$ and $x_1=y$, so $(x_0, 0)\sim(x_1, 1)$ with $x_0>x_1>0$. By A5, A2 and the 0 part of A3, $(x_1, 2)\sim(x_0, 1)>(x_1, 1)>(0, 1)\sim(0, 2)$, so $(x_1, 2)>(x_1, 1)>(0, 2)$. Hence, by A7, there is an x_2 with $x_1>x_2>0$ such that $(x_2, 2)\sim(x_1, 1)$. Continue in this manner to get a decreasing sequence $x_0, x_1, x_2,\ldots$ of positive x_i such that

$$(x_0, 0) \sim (x_1, 1) \sim (x_2, 2) \sim (x_3, 3) \sim\ldots.$$

Let $c=\inf\{x_0, x_1,\ldots\}$. By A7, $u(c, k)=\inf\{u(x_{i+k}, k)\colon i=0, 1,\ldots\}$ for any k in $\{0, 1, 2,\ldots\}$. In addition, A5 and $(x_{i+k}, i+k)\sim(x_i, i)$ imply that $(x_{i+k}, k)\sim(x_i, 0)$. Therefore, for each $k\geq 0$, $u(c, k)=\inf\{u(x_i, 0)\colon i=0, 1,\ldots\}$. Define u_0 by

$$u_0 = u(c, 0) = u(c, 1) = u(c, 2) = \ldots,$$

and choose ε so that $0<\varepsilon<[u(x_0, 0)-u_0]/2$. Then for every $\gamma>0$ there is an i

such that $x_i - c < \gamma$ and $u(x_i, i) = u(x_0, 0) > u_0 + \varepsilon$. But this contradicts A7. Q. E. D.

Bell Laboratories, Inc., Murray Hill, New Jersey

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