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Multidimensional Reasoning in Games: Framework, Equilibrium, and Applications†

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We develop a framework for analyzing multidimensional reasoning in strategic interactions, which is motivated by two experimental findings: (i) in games with a large and complex strategy space, players tend to think in terms of strategy characteristics rather than the strategies themselves; (ii) in their strategic deliberation, players consider one characteristic at a time. A multidimensional equilibrium is a vector of characteristics representing a stable mode of behavior: a player does not wish to modify any one characteristic. The concept is applied to several economic interactions, where a vector of characteristics, rather than a distribution of strategies, is identified as stable. (*JEL* C72, D11, D91)

The starting point of this paper is the recognition that in games in which the space of strategies is large and complex players deliberate over the space of characteristics of strategies rather than over the strategies themselves. In Arad and Rubinstein (2012), we studied a version of the Colonel Blotto game. In this game, each player (in the role of a colonel) allocates 120 troops across 6 battlefields ordered in a line. We argued that the main dimensions considered by the subjects were:

(i) the number of reinforced battlefields, i.e., those with more than 20 troops,

(ii) the location of the reinforced battlefields (for example, the outer versus the inner battlefields), and

(iii) the choice of the unit digit in the number of troops on each battlefield: for example, whether to allocate 0, 1, or 2 on a battlefield that the player is prepared to abandon.

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Arad and Penczynski (2018) investigated the Blotto game as well as multi-object auctions (all-pay and winner-pay) using an experimental protocol that allows eliciting the subjects' strategic considerations: subjects played in teams and were allowed to communicate before making a final decision. The analysis of the written messages revealed that almost all of the subjects thought in terms of properties of strategies, rather than in terms of strategies, and thus demonstrated the prevalence of multidimensional reasoning.

These findings suggest that in complex games, one should look for regularity in the combination of chosen characteristics of strategies rather than in the particular strategies played (see also Harstad and Selten 2013, 2016). Formalizing such a notion of stability requires a new modeling approach. We suggest such an approach below.

In the proposed model, each player classifies the strategies along a number of dimensions. He makes a decision in each dimension based on the profile of characteristics (not strategies) that he believes the other player will choose. Once he has decided on the desirable characteristics of his strategy, he picks a strategy that has them all.

This two-stage process is a common real-life phenomenon. It is often the case that decision makers first decide on the principles of their plan of action and only then fill in the details necessary to implement the plan.

For simplicity, we confine the analysis to symmetric two-player strategic interactions. The model, which we call an *edited game*, extends the standard model of a game by including a specification of an array of dimensions. Each strategy has a number of characteristics, one in each dimension. The set of all strategies is partitioned by the characteristics they have in the various dimensions. A cell is a set of strategies that share characteristics in all dimensions. A candidate for equilibrium according to our proposed solution concept is a cell. A cell is unstable if changing one characteristic while keeping the others fixed is desirable in the following sense: there is a strategy that differs in its characteristics from those in the cell only in that one dimension and that performs better (than all the strategies in the cell) against the uniform distribution over the strategies in the cell.

The proposed solution concept, which we call *MD-equilibrium*, is a cell that is not unstable in the aforementioned sense. In other words, the cell contains an optimal strategy against the uniform distribution over the cell from among all those that do not differ from the cell in more than one dimension. Note that the definition of an MD-equilibrium does not rule out the possibility that there are better strategies that differ in more than one dimension from the strategies in the cell. If there are no such strategies, then we refer to the MD-equilibrium as *global*.

Thus, an MD-equilibrium provides only a rough prediction: it does not specify which strategies will be chosen but rather points to a collection of strategies that share a profile of characteristics, one for each dimension.

Like Nash equilibrium, an MD-equilibrium is a stable mode of behavior where there is no possibility of a profitable deviation by an individual player. Such a concept is appropriate for situations in which players have accumulated experience in playing the game and have settled on a particular mode of behavior. For the degenerate case of only one dimension and in which each cell consists of one strategy, an MD-equilibrium is identical to a symmetric pure strategy Nash equilibrium. Another degenerate case, which is discussed in Section V, is an *edited product game* in which there are a number of dimensions but a cell still consists of only one strategy. In this case, an MD-equilibrium is identical to a symmetric pure strategy Nash equilibrium of teams, where each team member is responsible for the choice in one dimension. However, the main interest in MD-equilibrium is in the non-degenerate cases, where each cell includes a number of strategies. In those cases, deviations are based on reasoning in terms of cells rather than strategies and therefore, the existence of MD-equilibrium does not follow from the existence of Nash equilibrium.

The purpose of the following example is to clarify the model and our solution concept.

Example: Consider the following edited "settlements" game. There are two players and five territories labeled $1, 2, 3, 4, 5$. A territory i has a value of i for a player who settles there. A player chooses either one territory or a pair of territories to settle in. In the case that the territories chosen by the two players coincide or overlap, there is a confrontation and neither of them gets to settle in any territories. In the case that the choices don't coincide or overlap, each keeps the territory or territories he chose and his utility is the sum of the values of those territories.

For completing the description of the edited game, suppose that each player has in mind two dimensions: the number of territories he wishes to settle in (i.e., one or two territories) and whether he chooses at least one of the low-valued territories (i.e., Territory 1 and Territory 2) or avoids them. Given these dimensions, the strategic considerations in choosing a cell are as follows: (i) settling in one territory decreases the probability of a confrontation but reduces the payoff in the case of no confrontation, and (ii) avoiding the low-valued territories increases the payoff in the case of no confrontation.

The following matrix presents the classification of the strategies in the edited game where each column corresponds to a characteristic in the first dimension while each row corresponds to a characteristic in the second:

	one	*two*
avoid (*low-valued_territories*)	3, 4, 5	3 & 4, 3 & 5, 4 & 5
not_avoid	1, 2	1 & 2, 1 & 3, 1 & 4, 1 & 5, 2 & 3, 2 & 4, 2 & 5

The cell $C_1 = (one, not\_avoid)$ is not an MD-equilibrium since any strategy in $(one, avoid)$ is better than any strategy in C_1 against the uniform distribution over C_1. The cell $C_3 = (two, avoid)$ is not an MD-equilibrium since the strategy 5 in $(one, avoid)$ yields an expected payoff of $5 \cdot (1/3)$ against C_3, while any strategy in C_3 yields 0 against C_3. The cell $C_2 = (two, not\_avoid)$ is not an MD-equilibrium because the best strategy in C_2 against C_2, 2 & 5, is not as good as the strategy 4 & 5, which yields higher payoff with higher probability.

The only MD-equilibrium is $C_4 = (one, avoid)$ where 5 attains an expected payoff of $5 \cdot (2/3)$ against C_4. This is not a global MD-equilibrium since the strategy 2 & 5 in (two, not_avoid) would yield a higher payoff of $7 \cdot (2/3)$ against C_4. This MD-equilibrium reflects a mode of behavior in which players strive to reduce the probability of a confrontation (by settling in one territory) while aiming for the higher valued territories (i.e., avoiding 1 and 2).

Note that the MD-equilibrium differs from all of the game's Nash equilibria. Any mixed strategy Nash equilibrium that contains in its support a strategy $x \in \{3, 4, 5\}$ must also contain the strategies 1 and 2 in its support since otherwise the strategy 2 & x or 1 & x would be strictly better than x.

It is worth emphasizing again that the MD-equilibrium concept involves the stability of characteristics of strategies rather than of the strategies themselves. When applying the model to an economic interaction, the modeler must specify the dimensions and their possible characteristics. The MD-equilibrium is of course sensitive to the specification of the dimensions, which is a merit of the model in our view. The way that a game is perceived by the players, and in particular how strategies are organized by dimensions, is an important part of the description of a strategic situation and might indeed affect behavior. The current model makes it possible to examine how different perceptions lead to different equilibria.

The modeler's choice of dimensions is subjective in the same manner that the choice of any other component of a game model is subjective. In modeling any strategic interaction, the modeler wishes to include only the relevant players and the relevant set of strategies and capture in the payoff function only the aspects most significant to the players. This requires that the modeler activate his judgment and his common sense and that he be familiar with the situation. The choice of dimensions is simply an additional element in the players' strategic reasoning that needs to be specified. When choosing this extra component, one can use introspection regarding the natural language used to organize the strategy space or turn to empirical evidence. We believe that there are some regularities in the dimensions perceived by players. This view is supported by Arad and Penczynski's (2018) experimental findings, according to which people have in mind similar dimensions in different resource allocation games.

After specifying the dimensions, the applied economist can use the solution concept to "predict" or explain the choice of certain types of strategies that share common features, without committing himself to a prediction of specific pure or mixed strategies. We find the MD-equilibrium therefore to be an attractive equilibrium concept, especially in the case of games with a large number of strategies that do not have a pure Nash equilibrium and in which the mixed strategy equilibrium is complicated (as in the Blotto game, multi-object auctions, and the two-dimensional Hotelling game analyzed in this paper).

The rest of the paper consists of three parts:

In the first part (Section I), we present and discuss the formal model and the solution concept.

In the second part (Section II–IV), we apply the concept to three examples of economic interactions. In each, we specify an edited game that consists of a specific array of dimensions. The three examples have a common feature in that a player has to allocate resources (troops or money) among a number of fronts (battlefields, auction items, or tennis courts). However, the examples differ in their stories and payoff functions. In addition, in order to illustrate the richness of the model, we intentionally choose different specifications of the dimensions and their characteristics in each example. The model is by no means limited to resource allocation games. We demonstrate the model's relevance in two additional types of games: the settlements game described above and Hotelling's spatial competition game discussed in Section V. We find it intuitive to apply the MD-equilibrium in games whose strategies can be described as an array of decisions. These include the following decisions: location and price (d'Aspremont, Gabszewicz, and Thisse 1979), research costs and legal costs (Tullock 1980), capacity and price (Kreps and Sheinkman 1983), and pricing and persuasive advertising (Bagwell 2007), among many other famous economic examples.

In the final part of the paper, we deal with two main issues: in Section V, we provide some existence results for MD-equilibrium and define a concept of mixed MD-equilibrium that exists for finite games. In Section VI, we extend the model to asymmetric games.

I. The Model and the Equilibrium Concept

A. *The Model*

The basic component of the model is a symmetric two-player game (S, u) where S is each player's action set and $u(s, s')$ is a player's payoff if he chooses s when his opponent chooses s'. Assume that the set S is finite or that the function u is continuous.

We depart from the standard model by adding a description of the players' perception of the space of the strategies. We have in mind that a player thinks about the strategies in terms of K dimensions. Each strategy has a characteristic in each of the dimensions. A profile of characteristics—one in each dimension—fits a set of strategies that share this combination. A player deliberates over the space of the vectors of characteristics rather than on the space of the strategies. This leads to the following definition.

DEFINITION 1: *An edited symmetric game is a tuple* $<S, u, (D_k)_{k=1,\ldots,K}>$ *where* (S, u) *is a symmetric game and each* D_k *is a function that assigns to every strategy* s *a characteristic* $D_k(s)$.

Each k stands for a "dimension." The symbol d_k denotes a generic characteristic of the kth dimension. The notation $d = (d_k)_{k=1,\ldots,K}$ is used for the set of strategies s for which $D_k(s) = d_k$ for all k. We call such a set a cell. Note that each function D_k partitions the strategies by their D_k-characteristic and the set of all cells is the join of all those K partitions.

It is straightforward to extend the definition to edited n-player asymmetric games (see Section VI). However, we choose to focus in the paper on symmetric edited games in order to keep the notation simple and in order to focus on the conceptual issues.

B. *The Solution Concept*

We seek a stability concept in the spirit of a symmetric Nash equilibrium, which is defined on the space of vectors of characteristics (cells) rather than on the space of strategies. In order to define a profile of characteristics as an equilibrium of an edited game, we introduce the concept of a *proper response*, which is analogous to that of a best response.

DEFINITION 2: *The characteristic d_k^* is a proper response in the kth dimension to the cell $d = (d_k)$, denoted by $d_k^* \in PR_k(d)$, if the cell (d_{-k}, d_k^*) contains a best response to the uniform distribution over d, when restricted to strategies in $\cup_{e_k}(d_{-k}, e_k)$.*

The definition captures the model's two key assumptions: a player thinks in terms of characteristics and considers one dimension at a time. Thus, a player views the characteristic d_k^* as a proper response in the kth dimension to a cell d if from among all strategies which share with d all characteristics besides that in the kth dimension, a best-response strategy to the uniform distribution over d has d_k^* as its kth characteristic. The existence of such an optimal strategy is a player's justification for choosing d_k^* in the kth dimension as a response to d.

This definition reflects one way in which to formalize the proper response mode of reasoning; other ways will be discussed in Section IID. We do not argue that players actually implement the calculation described in the formal definition of a proper response; rather, we view it as an approximation of the reasoning process in complex settings. In the analysis of families of edited games below, we will demonstrate that our concept of proper response often captures intuitive strategic considerations.

We now arrive at the definition of the solution concept.

DEFINITION 3: *An* MD-equilibrium (*Multidimensional equilibrium*) *of the edited symmetric game $<S, u, (D_k)_{k=1,\dots,K}>$ is a nonempty cell $d^* = (d_k^*)$ such that $d_k^* \in PR_k(d^*)$ for all k.*

Thus, a candidate for MD-equilibrium is a cell, i.e., a vector that specifies a characteristic in each dimension. A player considers the dimensions one at a time and finds each characteristic d_k^* to be a proper response to the equilibrium cell.

Denote by $u(s, C)$ the expected utility when playing the strategy s against the uniform distribution over the cell C. An equivalent definition of the MD-equilibrium is a cell $d^* = (d_k^*)$ satisfying that there is a strategy in the cell that maximizes $u(s, d^*)$ over the set of strategies that differ in at most one characteristic from the cell characteristics. In an MD-equilibrium, a player does not find any such strategy to be better than all strategies in the cell when playing against the uniform distribution over the cell.

The concept concerns only the players' choice of a cell and is silent about what brings a player to choose a particular strategy in the cell. In particular, it does not imply that the strategies in the equilibrium cell are chosen with equal probability nor that the player chooses the optimal strategy against this distribution. The uniform distribution as well as the optimal strategy serve as ingredients in the formalization of the players intuitive proper response reasoning process.

The concept of MD-equilibrium is a mix of categorical thinking and dimensional thinking, but one can think about the two aspects separately. Thus, one can focus only on dimensional thinking by applying the MD-equilibrium concept to *product games* where each strategy is a vector, a player decides on each component separately but each cell is a single strategy (see Section V). Alternatively, one might wish to focus only on categorical thinking and allow a player to reason in terms of characteristics of a strategy though he does so simultaneously for all dimensions. In that case, we can apply the following definition.

DEFINITION 4: *An MD-equilibrium is global if the equilibrium cell contains a strategy that is a best response to the uniform distribution over the cell from among all strategies in S.*

In other words, an MD-equilibrium is not global if some strategy, which differs from it in at least two dimensions, is a better response to the uniform distribution on the MD-equilibrium cell than all strategies in the cell itself.

C. *Comments about the Model and the Solution Concept*

(i) The MD-equilibrium concept is appropriate for situations in which players do not collect precise information about other players' behavior due to coarse observations, limited memory, receipt of only rough information from others, etc. In such situations, a player may become aware of the stability of behavior in terms of the strategies' characteristics even if he does not have reliable information on the distribution of chosen strategies. This is reflected in the idea that in MD-equilibrium, players hold correct beliefs on the chosen cell but not on the chosen strategies within the cell.

(ii) In his proper response calculation, a player assumes a uniform distribution over the strategies in the cell. Thus, it fits a scenario in which a player assumes that his opponent chooses some strategy in a given cell, but has no reason to believe that one strategy is more likely to be chosen than another. The approach is consistent with the "principle of indifference," which is used in, for example, the extensive literatures on risk dominance (see Harsanyi and Selten 1988) and on k-level reasoning (see Stahl and Wilson 1995).

(iii) In the economic examples that follow, we show that an MD-equilibrium exists. However, as in the case of a pure Nash equilibrium, an MD-equilibrium does not always exist. In Section V, we present an existence theorem for edited product games where the set of the strategies is a product set and each cell consists of exactly one strategy. We also prove an existence theorem for supermodular games where the strategies are classified along a unique dimension. Finally, we suggest a mixed strategy version of the solution concept and prove the existence of a mixed strategy MD-equilibrium in any finite edited game with a unique dimension.

(iv) Our solution concept can be expressed using the notion of Nash equilibrium. Consider the auxiliary "cells game" that is based on the edited game $<S, u, (D_k)_{k=1,\ldots,K}>$ and is defined as follows (for simplicity, assume $K = 1$): each player chooses a cell. A player's payoff if he chooses d and his opponent chooses d' is $\max_{x \in d} u(x, d')$, that is, the maximal possible payoff when choosing a strategy in d, given the belief that his opponent chooses each strategy in d' with equal probability. Thus, in the cells game, given a player's choice of a cell, he is assumed to choose different strategies within the cell against different cells of his opponent. Notice that the payoffs attached to a pair of cells might not be feasible in the original game (S, u). Our solution concept is any symmetric Nash equilibrium of the auxiliary "cells game." However, we do not find this connection to be helpful in characterizing the MD-equilibrium, and in Section VC we explain why the cells game is not an appropriate basis for defining a reasonable "mixed" version of the MD-equilibrium.

(v) The experimental evidence that motivated this paper (Arad and Rubinstein 2012 and Arad and Penczynski 2018) suggested that players think in terms of characteristics of strategies rather than in terms of strategies. However, these experiments were not designed to test the MD-equilibrium "predictions" because they involved a single play of games in which stability is usually not expected. To test the MD-equilibrium (or the Nash equilibrium for that matter) players must be allowed to accumulate experience by playing the game multiple times.

(vi) The MD-equilibrium prediction depends crucially on the specification of dimensions and their possible characteristics since it provides the language used to define the set of cells that are the candidates for MD-equilibrium. Furthermore, even if the same set of cells is derived from two different specifications of dimensions, the permissible deviations may differ and hence the MD-equilibria will not necessarily be identical.

(vii) Some of the ingredients of the MD-equilibrium concept are shared by other economic concepts. In particular:

(a) There are several game-theoretic models in which the solution concept is a set of strategies rather than a single strategy. In particular, see Basu and Weibull (1991) who (adjusted for the symmetric case) search for minimal sets of strategies for which all best responses to any belief on the set are inside the set (see also the discussion in Myerson and Weibull 2015, 950).

(b) Previous examples of categorical beliefs (i.e., players' beliefs that are framed in terms of categories rather than strategies) include numerous models with analogy-based reasoning due to Philippe Jehiel (see, for example, Jehiel 2005) as well as Piccione and Rubinstein (2003).

(c) A player in our model can be thought of as a team in the sense of Marschak and Radner (1972), such that all members share the same target and each is responsible for choosing one characteristic of the team's decision. (See also Guney and Richter 2016 for the related concept of D-Nash equilibrium). In a standard team game, an array of characteristics defines a unique strategy. In our setup, a manager collects members' choices and chooses one of the strategies that is consistent with the chosen array of characteristics.

D. *Alternative Solution Concepts*

As emphasized above, we are not claiming that the solution concept is the only or best way to construct an equilibrium concept in a strategic situation in which players think in terms of characteristics rather than strategies. The definition of the MD-equilibrium relies on the notion of proper response. Any alternative definition of the proper response operator will induce an alternative solution concept.

This paper's approach is that a player finds a deviation from A to B to be profitable, given A, if B contains a strategy that achieves a higher expected payoff than any strategy in A against a uniform distribution over the strategies in A.

Following are two reasonable alternatives that differ in the circumstances under which a player finds it profitable to deviate from cell A to cell B, given that he expects the other player to choose a strategy in A:

(i) The representative strategy approach: given the expectation that the other player will choose a strategy in the cell A, a player anchors his deliberation on the strategy a^*, which he finds to be the representative strategy of the cell A. He looks for the best responses to a^* in A and in B, i.e., a and b, respectively. He finds the deviation from A to B profitable if b yields a higher payoff than a against a^*.

Note that unlike this paper's approach, this approach requires adding a criterion to the edited game that will determine the representative strategy within a cell. For example, in some contexts, one can think of the most salient or aesthetic strategy within a cell as being representative. In other contexts, especially when the cell contains an interval of numbers, a representative strategy might be the middle point of the cell.

(ii) The Uniform versus Uniform approach: Define $U(X,Y)$ to be the expected payoff of a player who randomizes uniformly over X given that his opponent randomizes uniformly over Y. The player deviates from A to B given that the other player plays A if $U(A,A) < U(B,A)$. Thus, a choice of an array of characteristics, i.e., of a cell, is equivalent to a choice of a uniform distribution over the chosen cell.

This approach can be interpreted to mean that a player believes his opponent's choice will be in a certain cell but he does not have any belief regarding which strategy within the cell he will choose. Therefore, he treats all the strategies in the opponent's cell as equally likely. Furthermore, when considering a cell to choose a strategy from, the player is unsure about the particular strategy he himself will choose within the cell, and hence, he again considers all the strategies in the cell as equally likely. Thus, the player is equally uncertain about his opponent's strategy as he is about his own.

We find this paper's approach more appropriate than approach (ii) in many circumstances since it makes sense to assume that a player has more vagueness about his opponent's strategy within a cell than about his own. It is likely that when deliberating about his choice of a cell, a player has in mind concrete actions.

II. Colonel Blotto: Reasoning on the Number of Reinforced Battlefields and Their Location

In this section, we apply the solution concept to a variant of the famous Colonel Blotto game, which originally appeared in Borel (1921). In this variant, there are two generals and each has N troops at his disposal. (For simplicity, we confine ourselves to values of N that are multiples of 6.) Each general allocates his troops among three battlefields denoted 1, 2, and 3 (we refer to field 2 as the center and to fields 1 and 3 as the edges). The set of strategies for each general is $S = \{(x_1, x_2, x_3) \,|\, x_i$ is a nonnegative integer and $\sum_{i=1,2,3} x_i = N\}$. The set contains $\frac{(N+2)!}{N! \cdot 2}$ strategies. When a player uses a strategy $x = (x_1, x_2, x_3)$ against a strategy $y = (y_1, y_2, y_3)$, he scores one point in field i if $x_i > y_i$, half a point if $x_i = y_i$, and 0 otherwise. His score is the sum of the points he scores in the three fields. A match between two strategies can yield only one of three scores: 2:1 (a win), 1.5:1.5 (a draw), 1:2 (a loss). Each general wishes to maximize his expected number of points.

The Blotto game has received widespread attention due to its interpretation in the political economics literature as a game between two presidential candidates who allocate their limited budgets among campaigns in the "battlefield" states (see, for example, Brams 1978). Myerson (1993) suggested an alternative interpretation of the Blotto game as a vote-buying game. The game can also be interpreted as an R&D race between two firms who compete by allocating their limited resources among a number of projects.

As mentioned in the introduction, experimental evidence indicates that the Blotto game triggers multidimensional reasoning. In particular, the vast majority of players choose their strategy after deliberating on the number of fields in which to concentrate their resources. We define the number of reinforced fields as the first dimension in the edited game.

Experimental results also suggest that participants take into account the order of the battlefields. Therefore, the second dimension is defined as the ordering of the three divisions. In other words, does the player assign the divisions in increasing order, in decreasing order, or not according to any of the two orderings?

Formally, the edited game involves the following two dimensions:

(i) The number of fields (1, 2, or 3) in which the player reinforces his troops, where field i is reinforced in x if $x_i \geq N/3$.

(ii) The order of the troop assignments: this dimension receives the characteristic "$\nearrow$" if the assignments are in increasing order ($x_1 \leq x_2 \leq x_3$ with at least one strict inequality); "$\searrow$" if the assignments are in decreasing order ($x_1 \geq x_2 \geq x_3$ with at least one strict inequality). Otherwise, it gets the characteristic "*other*."

To illustrate, following is the classification of the 28 strategies for the case of $N = 6$ (the notation *abc* stands for the strategy (a, b, c)):

	1*st dimension*		
2nd dimension	1	2	3
decreasing $\searrow$	411, 510, 600	321, 330, 420	–
increasing $\nearrow$	006, 015, 114	024, 033, 123	–
other	051, 060, 105 141, 150, 501	**042**, **132**, **204**, **213**, **231** **240**, **303**, **312**, **402**	222

The cell $(1, \searrow)$, for example, is not an MD-equilibrium since $\searrow$ is not a proper response in the second dimension to $(1, \searrow)$: the best strategy against $(1, \searrow)$ within the cell is $(4, 1, 1)$ (which achieves one win and two draws against the three strategies in $(1, \searrow)$, while $(0, 5, 1) \in (1, other)$ does better with two wins and one draw). Intuitively, abandoning the first field, reinforcing the second, and placing a positive number of troops on the third guarantees scoring a point on the second field and often scoring a point on the third, which is the weakest of the $(1, \searrow)$ strategies.

The cell $(2, other)$ is a (global) MD-equilibrium. Several strategies, including $(3, 0, 3) \in (2, other)$, are optimal against $(2, other)$ with an expected score of 1.72 (4 wins and 5 draws against the 9 strategies in $(2, other)$). This is higher than the score of any strategy outside the set, which at most achieves an expected score of 1.56 (for example, $(0, 2, 4)$ yields 4 wins, 2 draws, and 3 losses).

PROPOSITION 1: *In the edited Blotto game with* 3 *battlefields and N as a multiple of* 6, $(2, other)$ *is the only MD-equilibrium and is global.*

PROOF:

See Appendix.

The proof that $(2, other)$ is the only MD-equilibrium can be shown intuitively by means of the proper responses in the two dimensions.

Note that "*other*" is the union of the "peak $\cap$" strategies $(x_2 > x_1, x_3)$ and the "sink $\cup$" strategies $(x_2 < x_1, x_3)$.

$PR_1(2, other) = 2$: In any cell, there is a strategy that scores at least 1.5 points against the cell. However, any strategy that reinforces 1 or 3 battlefields scores less than 1.5 points: any strategy reinforcing only one battlefield scores one point against a strategy in $(2, other)$ if the reinforced battlefield is matched against a non-reinforced one, and scores at most 1.5 points on average otherwise. The single strategy of three reinforced battlefields typically wins on only one of them when playing against $(2, other)$.

$PR_2(2, other) = other$: The set $(2, other)$ is evenly partitioned into four classes of strategies denoted by $(very\_high, low, high)$, $(high, low, very\_high)$, $(low, very\_high, high)$, and $(high, very\_high, low)$, where $(very\_high, low, high)$, for example, includes all strategies in which the highest assignment is in Battlefield 1, the other reinforced battlefield is Battlefield 3, and the lowest assignment is in Battlefield 2. A strategy of the type $(very\_high, low, high)$ scores on average about 1.5, 1.5, 1.5, and 2 points against these four classes, respectively. However, a strategy in $(2, decreasing)$, which is of the type $(very\_high, high, low)$, scores on average only about 1.5, 2, 1, and 1.5 points against these four classes, respectively. The key deficiency of decreasing strategies is that they waste a medium-size assignment (in Battlefield 2) against the highest assignment in the peak strategies. A similar argument implies that a strategy in $(2, increasing)$ is inferior to a sink strategy of the type $(high, low, very\_high)$ when playing against the strategies in $(2, other)$.

$PR_1(1, other) \neq 1$: If a player believes that his opponent is concentrating his troops on one field, then splitting his troops among all three fields $((3, other))$ will yield a win for certain.

$PR_1(3, other) \neq 3$: Concentrating on the two edges and sacrificing the center $((2, other))$ will win against the strategy that splits the troops equally among the three fields.

$PR_2(2, \nearrow) \neq \nearrow$: A player can find a peak strategy with a medium size assignment to the first field and a high assignment to the second, while sacrificing the third field $((2, other))$. This will win against most strategies of increasing order. Similarly $PR_2(2, \searrow) \neq \searrow$.

$PR_1(1, \nearrow) \neq \nearrow$ since $(1, other)$ contains a peak strategy that abandons the third field, always wins the second, and has a good chance of winning the first. Similarly $PR_1(1, \searrow) \neq \searrow$.

Discussion.—The cell $(2, other)$ appears to be immune to deviations when applying intuitive coarse thinking, which is supported by fine calculations. This result suggests that the following coarse rule of behavior is stable: reinforce two battlefields and do not use a monotonic ordering for the three single-field assignments.

Interestingly, the MD-equilibrium in this game is in the spirit of the most successful strategies observed in Arad and Rubinstein's (2012) experiment of the one-shot Blotto game: these strategies reinforced two-thirds of the battlefields and did not use monotonic ordering.

Comparison to Nash Equilibrium.—The equilibrium of the Blotto game's continuous version was characterized by Roberson (2006), while that of the discrete version was characterized by Hart (2008). Both concluded that in equilibrium, players treat the battlefields symmetrically and the marginal distribution of the troops among the battlefields is essentially uniform in the (continuous or discrete) interval $[0, 2N/3]$.

The MD-equilibrium cell is very different from the support of the mixed strategy Nash equilibrium. In addition to the order feature (i.e., no increasing or decreasing strategy is included in the MD-equilibrium), all of the strategies in the MD-equilibrium cell have two reinforced fields, while in Nash equilibrium the expected number of reinforced fields is 1.5. Most importantly, Nash equilibrium and MD-equilibrium reflect very different modes of behavior. Whereas a player in MD-equilibrium makes a strategic decision to reinforce two battlefields and avoid a monotonic ordering, the behavior of a player in Nash equilibrium is consistent with the heuristic of maximizing the uncertainty regarding the number of troops in each battlefield.

III. A Three-Object All-Pay Auction: Thinking in Terms of Categorical (High/Low) Bids

The following game is inspired by Rosenthal and Szentes (2003). There are three objects up for sale and two bidders. The bidders are expected-payoff maximizers and each receives a payoff M if he wins any two of the objects. No additional benefit is obtained from winning a third object and there is no benefit from winning only one object. Each bidder is allowed to make three bids. He pays what he bids for an object regardless of whether his is the winning bid. For each object, the highest bid wins and a tie is broken randomly. In the case that no one bids on an object, each bidder receives the object with probability $1/2$.

A strategy is a triple of three nonnegative integers, each between 0 and T where T is an even number. We denote a strategy of three bids by $x = (x_1, x_2, x_3)$. However, in this section, we have in mind a situation in which the objects are not naturally ordered. The player chooses three bids and assigns them randomly to the three objects. That is, if he chooses a triple (x_1, x_2, x_3), then with probability $1/6$ he bids x_1 on object $\sigma(1)$, x_2 on object $\sigma(2)$, and x_3 on object $\sigma(3)$, where σ is any permutation of $\{1, 2, 3\}$.

For simplicity, we exclude $(0, 0, 0)$ from the set of strategies and focus on parameters of the model (i.e., M and T) for which the equilibrium expected payoff is nonnegative.

The game has two main interpretations: that of a multi-object auction where the bidders' interest in one object depends on the availability of another (such as in the case of oil leases and spectrum licenses); and that of an election game in which each of two candidates seeks to win a majority of districts and the votes a candidate receives in each district depend on the relative investment of the two candidates in the district. We believe that in such circumstances even sophisticated players think in terms of categories. For example, a player in the election game might decide on the number of districts in which he will spend his campaign funds and the relative amounts he will spend in those districts (high amounts in all districts, low amounts in all districts, or a mix of low amounts in some districts and high amounts in others).

We construct an edited game that can be used to determine whether there are stable descriptions of behavior in which: (i) players choose whether to bid on all the objects or only on a partial set of objects (without explicitly deciding on which objects), and (ii) players decide whether to place a high or low bid on each object (without deciding on the exact size of each bid). Experimental evidence for multi-object auctions suggests that both these dimensions are frequently considered by players (Arad and Penczynski 2018).

Formally, we specify the dimensions as follows:

(i) The number of positive bids, which can take a characteristic of either 1, 2, or 3.

(ii) The mix of high and low bids (among the positive bids). We divide the set of nonzero bids into $Low = \{1, \ldots, T/2\}$ and $High = \{T/2 + 1, \ldots, T\}$. It is assumed that this dimension can receive three characteristics: "L" (all positive bids are in *Low*), "*mix*" (at least one high and one low bid), and "H" (all positive bids are in *High*). Thus, for example, in the case of $T = 4$ the strategies $(1, 4, 4)$ and $(2, 3, 1)$ are in $(3, mix)$, the strategy $(1, 2, 0)$ is in $(2, L)$, and the strategy $(0, 0, 4)$ is in $(1, H)$.

We now characterize the MD-equilibria for a range of parameters in which the prize is large enough to justify choosing the highest possible bid on two objects for some beliefs, and small enough such that it is not always beneficial for a player to increase his bid by one unit if this has a positive effect on the probability of winning. In this domain, a player faces a real trade-off between decreasing the costs of the bids and increasing the probability of winning the prize.

PROPOSITION 2: *In the edited* 3*-object two-bidder simultaneous all-pay auction with* $2T < M < T^2/2$, *the cells* $(3, mix)$ *and* $(2, H)$ *are the only MD-equilibria. The cells are non-global unless* $\frac{3(T-2)}{T-1} > \frac{M}{T} > \frac{3T}{T-1}$ (*that is, with the exception of the case in which* M/T *is around* 3).

PROOF:

We focus here on the intuitive strategic considerations and direct the reader to the Appendix for further details.

Notation: For a cell C and a strategy $x = (x_1, x_2, x_3)$, denote by $W(x, C)$ the probability that the strategy wins M against the uniform distribution over C. The marginal increase in the probability of winning M by adding one unit of investment to the first component of the strategy is denoted by $\Delta(x, C) = W((x_1 + 1, x_2, x_3), C) - W((x_1, x_2, x_3), C)$. The three components are symmetric and thus the calculation of the marginals on one component is valid for the others as well. A player's expected payoff from playing x against the uniform distribution over C is $u(x, C) = W(x, C)M - \sum_{i=1,2,3} x_i$. Let $\Pr_C$(*statement* about a strategy x) denote the proportion of strategies in C satisfying the statement.

The following lemma provides an explicit expression for $\Delta((x_1, x_2, x_3), C)$.

LEMMA 1: *For a cell C and a strategy (x_1, x_2, x_3)*: $\Delta((x_1, x_2, x_3), C) = \Pr_C(y_1 \in \{x_1, x_1 + 1\}$ *and* $(y_2 = x_2$ *or* $y_3 = x_3))/4 + \Pr_C(y_1 \in \{x_1 + 1, x_1\}$ *and either* $y_2 > x_2$ *and* $x_3 > y_3$ *or* $y_2 < x_2$ *and* $x_3 < y_3)/2$.

PROOF:

See Appendix.

A strategy is said to be an *edge* strategy if all of its positive components are on the edges of the categories *Low* and *High*, that is within the set $\{1, T/2, T/2 + 1, T\}$. Let $edge(C)$ be the set of edge strategies in C. Lemma 1 implies Lemma 2, which states that the maximization of the expected payoff over cell C_1 below, while playing against the uniform distribution on a cell C_2, has a solution on the edge of C_1.

LEMMA 2: *For any two cells C_1 and C_2, the maximization* $\max_{s \in C_1} W(s, C_2)$ *has a solution in $edge(C_1)$.*

PROOF:

Consider three numbers $t, t+1, t+2$ belonging to the same category, either *Low* or *High*. The events presented in the above table for $x_1 = t$ and $x_1 = t + 1$ have the same probability. Thus, by Lemma 1, $\Delta((t + 1, x_2, x_3), C) = \Delta((t, x_2, x_3), C)$. Therefore, the maximization $\max_{s \in C_1} W(s, C_2)$ must have a solution that is an edge strategy. ∎

The following three claims show that only $(2, H)$ and $(3, mix)$ are MD-equilibria.

CLAIM 1: *All the cells besides $(2, H)$ and $(3, mix)$ are not MD-equilibria.*

The following are intuitive considerations, which show those cells to be unstable.

$PR_1(1, L) \neq 1$: The strategy $(1, 1, 1) \in (3, L)$ always wins against $(1, L)$ with almost no cost; in contrast, a strategy in $(1, L)$ wins with probability of at most $7/12$ against $(1, L)$.

$PR_1(1,H) \neq 1$ or $PR_2(1,H) \neq H$: There are two candidates for best strategy in $(1,H)$ when playing against $(1,H)$. The strategy $(T,0,0)$ wins with probability of about $7/12$ against $(1,H)$ but costs T and is inferior to $(T/2+1, T/2+1, T/2+1) \in (3,H)$, which costs only about $T/2$ more but adds $5/12$ to the chances of winning. The strategy $(T/2+1,0,0)$ is clearly inferior to $(1,0,0) \in (1,L)$ since using the lowest bid in H is a waste against $(1,H)$.

$PR_1(3,H) \neq 3$: With the expectation that the other player will choose some three high bids, a player believes that two maximal bids $((2,H))$ are sufficient to almost ensure winning the two objects and saving the cost of one high bid.

$PR_1(3,L) \neq 3$ or $PR_2(3,L) \neq L$: If the best response to $(3,L)$ in $(3,L)$ involves only two assignments of $T/2$, then the third bid of 1 plays only a small role in determining the probability of winning and hence should be dropped (a strategy in $(2,L)$). If the best response to $(3,L)$ in $(3,L)$ happens to involve three assignments of $T/2$, then a strategy of the type $(1, T/2+1, T/2+1) \in (3, mix)$ guarantees winning with lower costs.

$PR_2(2,L) \neq L$: The assumption that $M > 2T$ makes $(T/2, T/2, 0)$ the best strategy within $(2,L)$ against the cell. However, in that case, a strategy $(0, T/2+1, T/2+1) \in (2,H)$ involves only a small additional cost and guarantees winning.

$PR_1(2, mix) \neq 2$: Any $(1, x_2, x_3) \in (3, mix)$ is superior to $(0, x_2, x_3) \in (2, mix)$ against $(2, mix)$ since it increases the probability of winning by at least $1/12$ and involves only a negligible additional cost.

CLAIM 2: *The cells $(2,H)$ and $(3,mix)$ are MD-equilibria. They are non-global unless* $\frac{3(T-2)}{T-1} > \frac{M}{T} > \frac{3T}{T-1}$.

PROOF:

The formal proof for $(2,H)$ appears in the Appendix. We present here only an intuitive proof for $(3,mix)$. All statements of the type "a strategy wins against $(3,mix)$ with probability α" should be read as "with probability of *approximately* α" and the symbol $\sim m$ stands for "m or $m+1$."

The strategy $(1,T,T)$ wins almost always against strategies in $(3,mix)$ and thus its expected payoff is about $M-2T$. We will show that the only edge strategy that does better than $(1,T,T)$ against $(3,mix)$ is $(0,T,T) \in (2,H)$, and hence $(3,mix)$ is a non-global MD-equilibrium.

Any edge strategy that costs $2T + T/2$ or more is inferior to $(1,T,T)$ against $(3,mix)$.

The edge strategies that cost $3T/2$ are $(\sim 0, \sim T/2, T)$ and $(\sim T/2, \sim T/2, \sim T/2)$, all of which win with probability $1/2$ against $(3,mix)$ and have an expected payoff of $M/2 - 3T/2$, which is smaller than $M-2T$.

The edge strategies that cost T are $(\sim 0, \sim T/2, \sim T/2)$, which win with probability $1/6$ (against all strategies in $High \times Low \times Low$), and $(\sim 0, \sim 0, T)$ which always lose. Thus, all these strategies yield an expected payoff of at most $M/6 - T < M - 2T$.

Any edge strategy that costs $T/2$ or less always loses.

Finally, there are two types of edge strategies that cost $2T$: Any strategy of the type $(\sim T/2, \sim T/2, T)$ wins with probability $5/6$ and thus is inferior to $(1, T, T)$. The strategy $(0, T, T) \in (2, H)$ wins with probability 1 and its payoff evaluation requires a more precise calculation. It does (slightly) better than $(1, T, T)$ against $(3, mix)$ since

$$\Delta((0, T, T), (3, mix)) \;=\; \Pr_{(3, mix)}(y_1 = 1 \text{ and } (y_2 = T \text{ or } y_3 = T))/4 \;=\; \frac{(2T-1)}{6(T/2)^3}\frac{1}{4}$$

and

$$M\frac{2T-1}{24(T/2)^3} \;<\; 1 \quad \text{for} \quad M \;<\; T^2/2.$$

Claim 2 states that $(2, H)$ is an MD-equilibrium and that it is not global. The result captures a natural process of strategic deliberation in which a player who expects his opponent to choose two high bids believes that he will almost surely win if he makes two maximal bids. Reducing the number of bids to one will significantly undermine his chances of winning two objects. Making three high bids is wasteful since it will increase only marginally his chances of winning and involves a large additional cost. Making only two bids, with at least one of them low, dramatically reduces the chances of winning at least two objects.

Discussion.—We find that there are two MD-equilibria in this edited game. The first fits a norm of behavior according to which players bid high for two of the three objects only. The other involves bidding on all three objects such that at least one bid is low and at least one bid is high. The global best response to each of the MD-equilibria lies in the other equilibrium cell and the expected payoff for maximization against the equilibrium cell is almost identical, i.e., $M - 2T$.

The concept of MD-equilibrium captures strategic reasoning only in the dimensions that are specified in the edited game. Thus, in particular, it does not capture a player's natural consideration to deviate from the MD-equilibrium cell $(2, H)$ by increasing the bid on the neglected object from 0 to 1. Such a consideration can be captured by adding another dimension to the edited game, which could have the characteristics 0 or 1 according to the bid on a neglected object (and modifying the set L to include bids in $\{2, \ldots, T/2\}$). Indeed, Arad and Penczynski (2018) identified such a dimension in some participants' justification of their strategies in various multi-object auctions. However, they also found that a majority of the participants used only the unit digit 0 in bidding on the "neglected" objects and did not discuss this dimension in their justifications. Thus, it seems that most participants didn't think about this additional dimension, as assumed above.

Comparison to Nash Equilibrium.—The predictions of MD-equilibrium are different in nature from those of Nash equilibrium. Although we do not have a characterization of the Nash equilibria of this game, there is clearly no mixed strategy Nash equilibrium that is spanned by strategies in $(2, H)$ only. Given that the other player chooses strategies in $(2, H)$, then deviating to spending a very small amount on the third neglected object increases the probability of winning two objects dramatically.

Furthermore, the mixed strategy Nash equilibrium characterized in Rosenthal and Szentes (2003) for the continuous case (which probably approximates the equilibrium of this discrete game when the money grid is fine enough) is a uniform distribution over all strategies that lie on the surface of a specific tetrahedron. Its support includes strategies in all the cells except $(1, L)$ and $(1, H)$. Thus, the Nash equilibrium includes strategies from $(3, H)$ and $(3, L)$, which do not appear in either of the two MD-equilibria.

IV. The Tennis Coach Game with Costly Players: The Role of a Designated Location

The basic game in this example is an extension of Arad's (2012) tennis coach problem. Two tennis teams, each managed by a coach, compete on three courts denoted 1, 2, and 3. Each coach recruits his set of players and assigns a single tennis player to each court. Each tennis player has one of the skill levels $0, 1, 2, \ldots, T$. Thus, a coach's strategy is a triple (x_1, x_2, x_3), where $x_j \in \{0, 1, \ldots, T\}$. When a team $x = (x_1, x_2, x_3)$ plays against a team $y = (y_1, y_2, y_3)$, three matches are played: in court i the tennis player with skill level x_i confronts the player with skill level y_i. The team scores a point in each court i where its player is more skilled than his opponent $(x_i > y_i)$; it scores half a point if the two players are equally skilled $(x_i = y_i)$ and none if $x_i < y_i$. A tennis player of skill level x_i costs the coach cx_i where $c > 0$. Each coach faces a trade-off between performance and the cost of the players. The trade-off is inserted into the utility function as follows: $u(x, y) = |\{i | x_i > y_i\}| + |\{i | x_i = y_i\}|/2 - c\sum x_i$. Coaches maximize their expected utility.

The game involves a typical contest in which the outcome depends on costly investments made by the competitors. From a public welfare perspective, the investment would be considered a waste if the contest is viewed as pointless. The investment might be considered worthwhile if the public enjoys the contest and their enjoyment increases with the level of investment.

People often view locations asymmetrically even if there is no payoff-relevant difference between them. In other words, there is often a salient location, the assignment to which is viewed differently than the assignment to other locations. The standard solution concepts (and particularly the mixed strategy Nash equilibrium) treat all locations symmetrically. Some researchers, and in particular Bacharach (2006), have developed equilibrium concepts that take into account framing effects (for example, attraction to salience) in games where players make a decision involving locations (such as choosing a location or a subset of locations or deciding on the priority between locations).

We construct an edited game in which players perceive one of the courts as distinct from the others. We formalize this by specifying the characteristics in one of the dimensions to be whether the coach assigns the strongest tennis player on his team to the center court. Evidence that some people consider whether to focus on the center or the side locations has been found in experiments of related games (see Arad 2012). We will see that in an MD-equilibrium of the edited game, the strongest tennis player on the team will never be assigned to the center court.

Formally, the edited game consists of two dimensions:

(i) The sum of the skill levels of the three tennis players. A characteristic in this dimension can be any integer between 0 and $3T$.

(ii) Whether to assign the most skilled tennis player to the center court (namely, whether $x_2 = \max\{x_1, x_2, x_3\}$). This dimension can receive one of two characteristics: "strong" (having the most skilled player on the center court) and "not strong."

To illustrate, the following matrix presents the partition of the strategy space according to the two dimensions for the case of $T = 2$:

	0	1	2	3	4	5	6
strong	000	010	020, 011, 110	120, 021, 111	022, 220, 121	122, 221	222
not strong		100, 001	002, 101, 200	012, 201, 102, 210	211, 202, 112	212	

To make the situation nontrivial, we limit the range of c and T as follows:

(i) $1/2 > c$: That is, the coach is interested in increasing his expenses by one unit if it guarantees scoring half a point more.

(ii) $Tc > 1/2$: That is, it is not beneficial for the coach to spend the maximal possible amount in a court in order to earn half a point more than with no expenses.

The first and main part of Proposition 3 states that a cell of the type $(Q, strong)$ is never an MD-equilibrium. The intuition is as follows: the skill level assigned in this cell to each of the side courts cannot exceed that assigned to the center court and thus cannot be more than half of the total skill level. A coach can find a strategy against the cell that is superior to any strategy in the cell by using a strategy in $(Q, not\ strong)$ that abandons the center court and divides the total skill level as equally as possible between the two side courts. This strategy will score almost 2 points against $(Q, strong)$, while the maximal score obtained by a strategy in the cell when played against the cell is closer to 1.5 (though above).This result is independent of the parameters of the game.

Calculating the MD-equilibria among the cells of the form $(Q, not\ strong)$ depends on the values of c and T. For some parameters, no MD-equilibrium exists. The second part of Proposition 3 states that for values of c around $1/3$ the cell $(3, not\ strong)$ is an MD-equilibrium for all T. In this range, $(3, not\ strong)$ is the MD-equilibirum with the least cost. The cell $(1, not\ strong)$ is never an MD-equilibrium since $101 \in (2, not\ strong)$ outperform all strategies in the cell (that tie with all strategies in the cell). Similarly, the strategies in cell $(2, not\ strong)$ are outperformed by $110 \in (2, strong)$.

PROPOSITION 3: *In the edited tennis coach game with parameters T and c satisfying* $1/2 > c$ *and* $(3T - 1)c > 3/2$:

(*i*) *No cell* $(Q, strong)$ *is an MD-equilibrium.*

(*ii*) *If* $c \in (5/16, 3/8]$, *then the cell* $(3, not\ strong)$ *is a non-global MD-equilibrium for all T.*

PROOF:

See Appendix.

Discussion.—Given a decision about the total budget spent on the team, a coach naturally considers whether or not to assign the strongest player to the center court. A norm of assigning the strongest player to the designated court is not stable since a coach might think that if the other coach follows the norm, he can achieve almost 2 points without increasing the total cost of his team by putting his two strongest players on the edge courts. A stable mode of behavior must therefore include a norm not to put the strongest tennis player on the designated court.

Comparison to Nash Equilibrium.—The independence of the payoffs in each court and the additivity of the costs make it possible to calculate the mixed-strategy Nash equilibria of the game using the equilibria of the induced game in each single court. Consider the case of $T = 2$. The strategies in the auxiliary game are 0, 1, and 2, and the payoff matrix is given by:

	0	1	2
0	$1/2$	0	0
1	$1-c$	$1/2-c$	$-c$
2	$1-2c$	$1-2c$	$1/2-2c$

In the range of $1/2 \geq c \geq 1/4$, the auxiliary game has a unique mixed strategy Nash equilibrium $(1-2c, 4c-1, 1-2c)$ and the Nash equilibrium of the entire game will be any mixed strategy that induces this marginal distribution of skill levels in each of the three courts.

The strategies in the MD-equilibrium cell $(3, \textit{not strong})$ cannot span such a mixed strategy since none of the strategies in the cell use the skill level of 2 in the center court. Furthermore, for $1/3 > c$, the Nash equilibrium strategy uses extreme skill levels more than does the MD-equilibrium: the strength 1 appears in Nash equilibrium with less than half the probability of the appearance of a skill level within $\{0, 2\}$, whereas in any mixture of strategies from the MD-equilibrium cell $(3, \textit{not strong})$, the skill level 1 appears with at least half the probability of $\{0, 2\}$.

As mentioned above, the MD-equilibrium captures an intuitive consideration that is missing from the standard Nash equilibrium analysis: assigning the strong player to the center court is not stable since such a mode of behavior is easy to beat.

V. Existence: Product Games, Supermodular Games, and Mixed MD-Equilibrium

As in the case of Nash equilibrium in pure strategies, MD-equilibrium does not always exist. We present two examples of families of edited games in which the existence of MD-equilibrium under standard assumptions is guaranteed: (i) product edited games in which a strategy is a vector of K numbers and the dimensions are the K components of the vector, and (ii) edited two-player games with supermodular payoff functions. In addition, we extend the MD-equilibrium to a mixed MD-equilibrium and prove its existence in finite edited games.

A. *Product Edited Games*

The concept of MD-equilibrium includes two ingredients that differentiate it from standard Nash equilibrium:

(i) An MD-equilibrium is a set of strategies that share a specific characteristic in each dimension.

(ii) In looking for a best response strategy to a cell, a player considers only strategies that differ from the cell in at most one dimension.

The second ingredient can be discussed independently of the first for the family of edited games that we call *product edited games*. A product edited game is a tuple $<S, u, (D_k)_{k=1,\ldots,K}>$ for which the set of strategies S is a product set $S = \prod_{k=1,\ldots,K} S_k$, u is a payoff function, and $D_k(s) = s_k$. For product edited games, each cell is a singleton. A symmetric MD-equilibrium for product games is a strategy such that any deviation of a player in only one dimension (component) is not profitable. Note that if the payoff function is concave and differentiable, then the lack of profitable deviations in any dimension implies that other types of possible deviations are not profitable either. However, this is not the case for games in which the payoff function is not differentiable.

The following proposition states a simple condition that guarantees the existence of MD-equilibrium in product edited games.

PROPOSITION 4: *Let* $<S, u, (P_k)_{k=1,\ldots,K}>$ *be a product edited game where* S_k *is a closed interval of real numbers and* u *is a continuous function satisfying that for every* $(s_1, \ldots, s_K)$ *and every dimension* k, *the function* $f(y) = u((s_1, s_2, \ldots, s_{k-1}, y, s_{k+1}, \ldots, s_K), (s_1, \ldots, s_K))$ *is concave. Then, the edited game has an MD-equilibrium.*

PROOF:

Consider the correspondence $T : S \rightarrow S$ defined by

$$T(s_1, \ldots, s_K)$$

$$= \times_{k=1,\ldots,K} \Big\{ x_k \in S_k \,|\, x_k = \arg\max_y u\big((s_1, s_2, \ldots, s_{k-1}, y, s_{k+1}, \ldots, s_K), (s_1, \ldots, s_K)\big) \Big\}.$$

All sets in the range of the correspondence are products of closed intervals. By a standard fixed point argument, the correspondence has a fixed point that is an MD-equilibrium. ∎

To illustrate a product edited game, consider the following version of a two-dimensional Hotelling model: two political candidates are competing for votes. The set of policies S is the unit square and each policy represents a stand on two public issues. Each voter has an ideal point $h \in S$ and holds a strictly convex preference relation represented by a continuous function $u(s, h)$. The ideal points are distributed according to F. Each candidate selects a point in S. Voters maximize their utility, while candidates wish to maximize the number of votes they receive. A candidate considers each issue separately. This strategic situation can therefore be analyzed as a product edited game in which the two dimensions are the two issues.

This example is not covered by Proposition 4 since the candidates' induced payoff functions are not continuous. However, the following argument proves that an MD-equilibrium does exist. Consider a strategy (s_1, s_2). Each voter has a preferred point in $\{(s_1, y) \mid y \in [0,1]\}$. Let $m(s_1)$ be the median of those points and define $m(s_2)$ similarly. The function $M(s_1, s_2) = (m(s_2), m(s_1))$ has a fixed point, which is an MD-equilibrium by our definition. In the special case, in which the marginal distribution of preferred positions on one issue is independent of the preferred position on the other, the MD-equilibrium is the pair of medians of the two marginal distributions.

This argument was used in Roemer (2001, chapter 6) while proving that any Nash equilibrium in the above Hotelling game must be a fixed point of the function M. Roemer also showed that this point is generically not a Nash equilibrium of the Hotelling game, which implies that an MD-equilibrium of our edited game is generically not global.

B. *Existence of MD-Equilibrium in Supermodular Games*

The following is a simple example of an existence claim (suggested by Michael Richter) which relates to edited two-player games with supermodular payoff functions, in which the one-dimensional space of strategies is partitioned into a finite set of intervals.

PROPOSITION 5: *Let* $<S, u, \{D\}>$ *be an edited game where* $S = [m, M] \subset \mathbb{R}$, *and u is continuous and supermodular* (*in the sense of Topkins* 1979). *Assume that there is a sequence of points in* S, $m = a_0 < a_1 < \cdots < a_L = M$ *such that* $D(s) = l$ *if* $s \in P_l = [a_{l-1}, a_l]$. *Then, an MD-equilibrium exists.*

Comment.—Note that D is actually a correspondence in this case since a border point between two intervals receives two values. This is a straightforward extension of our model and solution concept.

PROOF:

By continuity, the best-response correspondence BR is well defined. Define the proper response correspondence PR by $l' \in PR(l)$ if there is $s \in P_{l'}$, which is a best response to the uniform distribution over P_l. Define the maximal proper response function by $MPR(l) = l'$ where l' is the highest index in $PR(l)$. We now show that MPR is a non-decreasing function. Suppose the contrary. Then, there are pairs $l_1 > l_1'$ and $l_2 < l_2'$ such that $MPR(l_2) = l_1$ and $MPR(l_2') = l_2'$. Therefore, there are $a \in P_{l_1}$ and $b \in P_{l_1'}$ such that $u(a, Unif(P_{l_2})) > u(b, Unif(P_{l_2}))$ and $u(b, Unif(P_{l_2'})) > u(a, Unif(P_{l_2'}))$, violating the supermodularity of u. Thus, the maximal proper response function is non-decreasing in l. A well-known result (which is a trivial case of Tarski's fixed point theorem) guarantees the existence of l such that $PR(l) = l$ and the cell P_l is an MD-equilbrium. ∎

C. *A Mixed MD-Equilibrium*

In this subsection, we define the concept of a mixed MD-equilibrium for a symmetric edited game with "one dimension," which is analogous to that of mixed strategy Nash equilibrium, and show that such an equilibrium always exists.

As discussed in Section IC, the MD-equilibrium is meant to capture situations in which players do not collect precise information about other players' strategies but rather about the characteristics of those strategies. A player will not deviate from a particular cell if he learns that the other players are playing strategies in that cell and the cell is a proper response to itself.

Similarly, one can think about stochastic stability. Players may realize that a number of cells are chosen by other players and that there is regularity in the frequencies of the chosen cells. The observed distribution of cells is stable if each cell in its support is a "proper response" to the distribution. Namely, a distribution of cells is an equilibrium if each cell in its support includes a strategy, which is a best response to the compound lottery that selects a cell and then uniformly selects a strategy in that cell. This definition reduces to the standard notion of symmetric mixed-strategy Nash equilibrium if each cell is a singleton and to the MD-equilibrium in the case that the distribution over the cells is degenerate.

Formally, let $<S, u, \{D\}>$ be a symmetric edited game with one dimension. Denote by V the range of the function D. We denote the cell of strategies $\{s \,|\, D(s) = v\}$ by (v). Define $\Delta = \Delta(V)$ to be the set of all lotteries on V. Thus, a member of Δ is a distribution over cells (for example, a player chooses some high price with probability $2/3$ and some low price with probability $1/3$). As in the case of standard mixed strategies, a member of Δ can be thought of as the belief of a player about the cells used by the other player. A candidate for equilibrium is a member of Δ. For $\pi \in \Delta$ to be an equilibrium, each cell in the support of π must be a proper response to the belief that the probability of the other player choosing (v) is $\pi(v)$ for each v. A distribution over cells $\pi \in \Delta$ is a mixed MD-equilibrium if every (v) for which $\pi(v) > 0$ contains a best response to the following compound lottery $\hat{\pi}$: the characteristic v is first selected with probability $\pi(v)$ and then, each strategy $s \in (v)$ is played with equal probability.

Our definition of mixed MD-equilibrium is distinct from the mixed-strategy Nash equilibrium of the auxiliary game defined in Section IC (comment (iv)). Note that the optimal strategy against the distribution of cells played by the opponent may differ from the strategies that are optimal against the individual cells. Thus, the implied (unreasonable) interpretation of the mixed-strategy Nash equilibrium for the auxiliary game is that a player who faces uncertainty regarding his opponent's choice of cells and assigns positive probabilities to a number of cells chooses the optimal strategy against each cell at the right moment. Our definition takes care of this bug.

PROPOSITION 6: *Any symmetric edited game with one dimension $<S, u, \{D\}>$ has a symmetric mixed MD-equilibrium.*

PROOF:

Define a correspondence $F : \Delta \rightarrow \Delta$ to be $\beta \in F(\alpha)$ if for every v in the support of β there is a strategy $s \in (v)$, which is a best response to $\hat{\beta}$. For every

α, the set $F(\alpha)$ is nonempty and convex. To see that F has a closed graph, consider a sequence (α^n, β^n) in the graph of F that converges to (α, β). If $\beta(v) > 0$, then for any n large enough $\beta^n(v) > 0$. Thus, the cell (v) contains a strategy s^n that maximizes $u(s, \widehat{\alpha^n})$. By the finiteness of S, there is a strategy s^* in (v) that maximizes $u(s, \widehat{\alpha^n})$ for an infinite number of ns. Since the function $u(s, \hat{\pi})$ is linear in π, s^* also maximizes $u(s, \hat{\alpha})$. Thus, $\beta \in F(\alpha)$ and the graph of F is closed. By Kakutani's fixed point theorem, F has a fixed point, which is a mixed MD-equilibrium. ∎

VI. Asymmetric Edited Games

The MD-equilibrium introduced in Section I for symmetric games can easily be extended to asymmetry between the players, whether due to differences in their sets of strategies or the payoff functions in the basic game or differences in the players' perceptions of the dimensions that partition the strategy space in the edited game.

An asymmetric edited game is a tuple $<S^i, u^i, (D^i_k)_{k=1,\ldots,K_i}>_{i\in N}$, where N is the set of players, S^i is player i's set of strategies, u^i is i's payoff function, and $(D^i_k)_{k=1,\ldots,K_i}$ is the collection of player i's dimensional functions. A cell for player i is a set of all strategies $s \in S^i$, which share the K_i characteristics $(D^i_k(s))_{1\leq k\leq K_i}$, that is, each cell is characterized by the choice of a characteristic in each of the K_i dimensions considered by i. An MD-equilibrium of the edited game is a profile of cells $(d^*(i))_{i\in N}$ such that for each i, a best response from among $\{s \in S^i \mid D^i_k(s) = d^*_k(i)$ for all k besides at most one dimension$\}$ to the uniform distribution over $\times_{j\neq i}(d^*(j))$ is in the cell $d^*(i)$. We have in mind that the edited game, including the different sets of dimensions used by each player, is common knowledge among the players.

As an example, consider a Blotto game with three battlefields and two players, B_6 and B_4. Player B_6 has 6 troops and B_4 has 4. Players have in mind two dimensions:

(i) The number of reinforced battlefields: a reinforced battlefield for player B_T includes at least $\lfloor T/3 \rfloor$ (that is, at least 2 troops for B_6 and at least 1 for B_4).

(ii) The order dimension: whether or not the order of the player's three troop assignments is monotonic.

Each of the following matrices represents one player's partition of his strategy space:

B_6	1	2	3
monotonic	411, 510, 600 006, 015, 114	321, 330, 420 024, 033, 123	– –
no order	051, 060, 105 141, 150, 501	**042**, **132***, **204**, **213**, **231** **240**, **303**, **312**, **402**	222

B_4	1	2	3
monotonic	400 004	**310***, **220** **022**, **013**	211 112
no order	040	202, 031, 103, 130, 301	121

One of the game's MD-equilibria is $((3, \textit{no order}), (1, \textit{monotonic})) = (\{222\}, \{400, 004\})$, which is not global because 411 is a better response to $\{400, 004\}$ for B_6.

The only other MD-equilibrium (marked in bold) is $((2, \textit{no order}), (2, \textit{monotonic}))$, which is global. The strategy 132 is B_6's best response to $(2, \textit{monotonic})$ with a payoff of 2.13. The structure of $(2, \textit{monotonic})$ for B_4 allows B_6 to win all battles in the center by deploying 3 troops there, which leaves him enough troops to win half of the other battles on the edges. The strategy 310 is B_4's best response to $(2, \textit{no order})$ with a payoff of 1.28.

Both MD-equilibria are consistent with the intuition that in a stable situation the weaker player focuses on a smaller number of fields than the stronger one in order to have a chance of winning at least in those fields despite his overall inferiority. Thus, in MD-equilibrium the weaker player uses an assignment of at least two troops in either one or two fields, whereas the stronger player assigns at least two troops to two fields according to all strategies in his MD-equilibrium cell.

The above feature of the MD-equilibrium concept is shared with the game's field-symmetric mixed strategy Nash equilibria. Consider the following auxiliary game where B_6's strategies are [600], [511], [411], [420], [330], [321], [222], while those of B_4 are [400], [310], [220], [211], where the meaning of $[abc]$ is that all its permutations will be played with equal probability:

	[400]	[310]	[220]	[211]
[600]	5/3	4/3	4/3	1
[510]	2	10/6	1.5	4/3
[411]	13/6	2	10/6	10/6
[420]	11/6	11/6	11/6	11/6
[330]	10/6	11/6	13/6	2
[321]	2	2	2	13/6
[222]	2	2	2	15/6

The value of the auxiliary game is 2, and in all Nash-equilibria the stronger player uses either [321] or [222] and the weaker player never uses [211]. Thus, in Nash equilibrium, the stronger player spreads his forces over the three fields (and reinforces two of them) while the weaker one places his troops on either one or two fields.

VII. Discussion

The paper introduces a concept of stability in the spirit of Nash equilibrium that applies to strategic situations in which a player's strategy space is large and complex and players reason in terms of characteristics of strategies rather than the strategies themselves. A symmetric MD-equilibrium is a stable mode of behavior in the sense that a player who considers deviating from it by altering his choice in one of the dimensions (while keeping the others fixed) will not find any justification to do so.

The MD-equilibrium "predicts" a profile of strategies' characteristics rather than the traditional prediction of a distribution of strategies. We find this approach to be more realistic in many circumstances. Thus, when asked about their strategy in a complicated strategic interaction, people do not usually describe a specific pure or mixed strategy. Rather, they tend to make some qualitative statement that sums up their strategy, such as: "I always bid high," "I concentrate my attention on only two fields of study," "I cooperate with only a few players," or "I am playing aggressively."

In order to apply the MD-equilibrium concept, one needs to specify how players characterize the strategies. However, the need for this additional information should not deter one from using the concept since the way in which players characterize strategies will influence their behavior and needs to be understood and integrated within the analysis. Adding this information provides a new tool for explaining the stability of certain modes of behavior in strategic situations.

The dimensions and characteristics of an edited game provide the language used to describe an MD-equilibrium. In each of the examples presented in the paper, we used a different language (i.e., specification of dimensions) and, therefore, the insights were expressed in the language of that specific example. Can we arrive at any general insights based on these examples? Overall, it can be said that a cell is not an MD-equilibrium if it is characterized by a regularity that can be used to find a strategy outside the cell that performs better against the cell than any strategy in the cell. For example, in the Blotto game discussed in Section II, the cell $(2, \textit{increasing})$ contains only strategies of the form $(\textit{low}, \textit{high}, \textit{very\_high})$ and most of those strategies are defeated by some strategy of the form $(\textit{high}, \textit{very\_high}, \textit{low})$ in $(2, \textit{other})$. In contrast, the MD-equilibrium $(2, \textit{other})$ contains an equal mixture of both peak and sink strategies and therefore is harder to beat. Similarly, in the auction game discussed in Section III, it is difficult to defeat the MD-equilibrium cell $(3, \textit{mixed})$ due to the uncertainty about the level of each of the bids. In the tennis game discussed in Section IV, the MD-equilibrium avoids the assignment of the strongest player to the center court, but does not specify where he should be assigned. The uncertainty regarding the location of the strongest player makes the cell unbeatable.

Our paper's journey began from experimental evidence that people think in terms of dimensions of strategies rather than the strategies themselves. These findings inspired us to formalize multidimensional reasoning and to suggest a framework based on the three concepts of "an edited game, a proper response, and MD-equilibrium," which is an expansion of "a game, a best response, and Nash equilibrium." We adopted a specific formal definition of a proper response. As mentioned, there are other reasonable definitions that yield different equilibrium concepts. The framework can also be adapted to nonequilibrium approaches. Overall, the paper is the first step toward a new paradigm for analyzing behavior in strategic interactions with a large and complex strategy space.

Appendix

PROPOSITION 1: *In the edited Blotto game with* 3 *fields and N, which is a multiple of* 6, $(2, \textit{other})$ *is the only MD-equilibrium and is global.*

PROOF:

Denote by $p(x, y)$ the number of points that a player using the strategy x scores against the strategy y. Given a cell C, let $p(x, C) = \sum_{y \in C} p(x, y)$. Obviously, comparing two strategies x and y played against the uniform distribution over C is equivalent to comparing $p(x, C)$ to $p(y, C)$. ■

CLAIM 1: *All cells besides* $(2, other)$ *are not MD-equilibria.*

$(1, \searrow)$ (and similarly $(1, \nearrow)$): The strategy $(0, 2N/3 + 1, N/3 - 1) \in (1, other)$ wins against all strategies in $(1, \searrow)$ except for $(N/3 + 2, N/3 - 1, N/3 - 1)$ with which it ties. Any strategy $(x_1, x_2, x_3) \in (1, \searrow)$ ties with itself and with at least one other strategy in $(1, \searrow)$: either $(x_1 + 1, x_2, x_3 - 1)$ (if $x_3 \geq 1$), or $(x_1 + 1, x_2 - 1, 0)$ (if $x_2 \geq 1$ and $x_3 = 0$) or $(N - 1, 1, 0)$ (if $x_2 = x_3 = 0$).

$(2, \searrow)$ (and similarly $(2, \nearrow)$): The strategy $(0, 2N/3, N/3) \in (2, other)$ wins against all strategies in $(2, \searrow)$, whereas any strategy in $(2, \searrow)$ ties with itself.

$(1, other)$: The strategy $(N/3, N/3, N/3) \in (3, other)$ wins against all the strategies in $(1, other)$.

$(3, other)$: The strategy $(N/2, 0, N/2) \in (2, other)$ wins against $(N/3, N/3, N/3)$, the only strategy in $(3, other)$.

CLAIM 2: *The cell* $(2, other)$ *is a* $(global)$ *MD-equilibrium.*

To illustrate, the following table presents the distribution of assignments in the center field and in an edge field for the case of $N = 18$ (where $(2, other)$ contains 51 strategies). The cell corresponding to assignment n and field i contains the number of strategies in $(2, other)$ for which the assignment in field i is n:

assignment	*field* 2	*category*	*field* 3	*category*
12	2	H_2	1	H_3
11	4	H_2	2	H_3
10	6	H_2	3	H_3
9	6	M_2	4	H_3
8	4	M_2	$5 + 2$	M_3
7	2	M_2	$6 + 4$	M_3
6	0	M_2	$6 + 6$	M_3
5	2	L_2	1	L_3
4	3	L_2	1	L_3
3	4	L_2	2	L_3
2	5	L_2	2	L_3
1	6	L_2	3	L_3
0	7	L_2	3	L_3

The pattern of the distributions is generalized in the following table (the explanations refer to six categories $H_2, M_2, L_2, H_3, M_3, L_3$ of pairs (n, i)):

	$x_2 = n \in$	# strategies in $(2, other)$	Explanation
H_2	$(N/2, 2N/3]$	$2(2N/3 - n + 1)$	x_1 or x_3 is in $[N/3, N - n]$
M_2	$[N/3, N/2]$	$2(n - N/3)$	x_1 or x_3 is in $[N/3, n - 1]$
L_2	$[0, N/3)$	$N/3 - n + 1$	edges are reinforced; $x_1 \in [N/3, 2N/3 - n]$

	$x_3 = n \in$	# strategies in $(2, other)$	Explanation
H_3	$[N/2, 2N/3]$	$2N/3 - n + 1$	$x_1 \in [N/3, N - n]$
M_3	$[N/3, N/2 - 1]$	$N - 2n + \min\{2N/3 - n + 1, N/3\}$	$x_2 \in [n + 1, N - n]$ or $x_2 \in [0, N/3 - 1] \cap [0, 2N/3 - n]$
L_3	$[0, N/3)$	$\lceil N/6 - n/2 \rceil$	$N/3 \leq x_1 < x_2$ and $x_1 \in [N/3, (N - n)/2)$

To see that $(2, other)$ is a global MD-equilibrium, we need to prove that for each strategy $x \notin (2, other)$, there is a strategy $y \in (2, other)$, which does at least as well as x against $(2, other)$.

Case 1: x has only one reinforced field.

If x has a peak in the center, it will lose against half of the strategies in $(2, other)$, which are "sinks," and thus its expected score is no more than 1.5. However, any set of strategies contains a strategy that achieves an expected score of at least 1.5 when playing against the set. Thus, there must be a strategy in $(2, other)$ that is weakly better than x. Otherwise, and without loss of generality, field 3 is reinforced. When playing against all strategies in $(2, other)$, x scores as follows: in field 3, at most $|(2, other)|$; in field 2, less than the sum of entries in category L_2; and in field 1, less than the sum of entries in category L_3. The sum of the entries in L_3 equals the sum of the entries in H_2:

$$\sum_{n<N/3} \lceil N/6 - n/2 \rceil = \sum_{N/2<n\leq 2N/3} 2(2N/3 - n + 1).$$

The sum of entries in M_2 is $2(\sum_{i=1,\ldots,N/6} i)$. Therefore,

$$p(x, (2, other)) < 2|(2, other)| - 2\left(\sum_{i=1,\ldots,N/6} i\right).$$

We will show that $p((N/2, 0, N/2), (2, other)) > p(x, (2, other))$. In each edge field, $(N/2, 0, N/2)$ wins a point against any assignment except those in H_3, where it loses to any assignment in $(N/2 + 1, 2N/3]$ (there are $\sum_{i=N/2+1,\ldots,2N/3} (2N/3 - i + 1) = \sum_{i=1,\ldots,N/6} i$ strategies with such assignments), and ties with the assignment $N/2$ (there are $N/6 + 1$ such strategies in

$(2, other)$). Furthermore, it scores half a point in the center when playing against an assignment 0 (there are $N/3 + 1$ strategies in $(2, other)$ with 0 in the center). Thus,

$$p((N/2, 0, N/2), (2, other))$$
$$\geq 2\left[|(2, other)| - \sum_{i=1,\ldots,N/6} i - (N/6 + 1)/2\right] + (N/3 + 1)/2$$
$$= 2|(2, other)| - 2\left(\sum_{i=1,\ldots,N/6} i\right) - 1/2.$$

Case 2: x has at least two reinforced fields.

We show that x is inferior to $(N/2, 0, N/2) \in (2, other)$.

Since $x \notin (2, other)$, it follows that the center must be reinforced and without loss of generality field 3 must also be, which implies that $x_2 \leq x_3$.

Consider a strategy x with $x_3 < N/2$ and therefore $x_1 > 0$. The strategy $(x_1 - 1, x_2, x_3 + 1)$ is superior to x against $(2, other)$ since the minimal marginal gain from adding 1 to field 3 (category M_3) is larger than the maximal marginal loss from subtracting 1 from field 1 (category L_3). Thus, x is inferior to a strategy of the type $(a, N/2 - a, N/2) \in (2, \nearrow)$.

Consider $x = (a, N/2 - a, N/2)$ where $a \leq N/6$. The strategy is not superior to $(0, N/2, N/2)$ because the sequence of marginal gains from increasing the assignment in field 1 $(N/6, N/6, N/6 - 1, N/6 - 1, N/6 - 2, N/6 - 2, \ldots)$ (category L_3) is dominated by the sequence of marginal losses from decreasing the assignment in the center $(N/3, N/3 - 2, N/3 - 4, \ldots)$ (category M_2).

Consider $x = (0, N/2, N/2)$. The strategy $(N/2, 0, N/2)$ is superior to x and its advantage is $p((N/2, 0, N/2), (2, other)) - p((0, N/2, N/2), (2, other)) = \sum_{i=1,\ldots,N/6} i + 1/2$. The number of assignments in $(2, other)$ that are greater than $N/2$ in field 2 is twice the number in field 3, which is $\sum_{i=1,\ldots,N/6} i$. The relative advantage of $(N/2, 0, N/2)$ over $(0, N/2, N/2)$, since it ties with the 0 assignment (the former strategy ties in the center and the latter ties in the edge), is larger than the disadvantage that it ties with the $N/2$ assignment. More formally, the total points scored by $(N/2, 0, N/2)$ in fields 1 and 2 when playing against all strategies in $(2, other)$ is $[|(2, other)| - \sum_{i=1,\ldots,N/6} i - (N/6 + 1)/2] + [(N/3 + 1)/2]$, while the analogous number of points for $(0, N/2, N/2)$ is $[(N/6)/2] + [|(2, other)| - 2\sum_{i=1,\ldots,N/6} i - (N/3)/2]$.

Finally, consider a strategy of the type $x = (a, N/2 - a - b, N/2 + b)$, where $a + b \leq N/6$ ($a \geq 0$, $b > 0$). The gain in field 3 from the extra b in the edge, compared to $(a, N/2 - a, N/2)$, is $[(N/6 + 1) + 2(N/6) + \cdots + 2(N/6 - b) + (N/6 - b + 1)]/2$ (category H_3), while the loss in the center from decreasing the assignment to $N/2 - a$ is at least b. Thus, for all b, we have $[(N/6 + 1) + 2(N/6) + \cdots + 2(N/6 - b) + (N/6 - b + 1)]/2 - b = (N/6 + 1)/2 - b - (N/6 - b + 1)/2 + \sum_{i=N/6-b+1}^{N/6} i = -b/2 + \sum_{i=N/6-b+1}^{N/6} i < \sum_{i=1,\ldots,N/6} i$. Therefore, x is inferior to $(N/2, 0, N/2)$. ∎

PROPOSITION 2: *For the edited* 3*-object two-bidder simultaneous all-pay auction with* $2T < M < T^2/2$, *the cells* $(3, mix)$ *and* $(2, H)$ *are the only MD-equilibria.*

The cells are non-global unless $\frac{3(T-2)}{T-1} > \frac{M}{T} > \frac{3T}{T-1}$ *(that is, with the exception of the case in which* M/T *is around* 3).

LEMMA 1: *For a cell* C *and a strategy* (x_1, x_2, x_3): $\Delta((x_1, x_2, x_3), C) = \Pr_C(y_1 \in \{x_1, x_1+1\}$ *and* $(y_2 = x_2$ *or* $y_3 = x_3))/4 + \Pr_C(y_1 \in \{x_1+1, x_1\}$ *and either* $y_2 > x_2$ *and* $x_3 > y_3$ *or* $y_2 < x_2$ *and* $x_3 < y_3)/2$.

PROOF:

The Lemma follows from the following table, which classifies all strategies y for which moving from (x_1, x_2, x_3) to (x_1+1, x_2, x_3) changes the probability of winning M when playing against y.

The event	Prob. of winning M increases				The increase
$y_1 \in \{x_1, x_1+1\}$ and ...	$y_1 = x_1$		$y_1 = x_1 + 1$		
	from	to	from	to	
either $y_2 > x_2$ and $x_3 > y_3$ or $y_2 < x_2$ and $x_3 < y_3$	1/2	1	0	1/2	1/2
either $y_2 > x_2$ and $y_3 = x_3$ or $y_2 = x_2$ and $y_3 > x_3$	1/4	1/2	0	1/4	1/4
both $y_2 = x_2$ and $y_3 = x_3$	1/2	3/4	1/4	1/2	1/4
either $y_2 < x_2$ and $y_3 = x_3$ or $y_2 = x_2$ and $y_3 < x_3$	3/4	1	1/2	3/4	1/4

LEMMA 2: *For any two cells* C_1 *and* C_2, *the maximization* $\max_{s \in C_1} W(s, C_2)$ *has a solution in edge*(C_1).

PROOF:

Consider three numbers $t, t+1, t+2$ belonging to the same category, either *Low* or *High*. The events presented in the above table for $x_1 = t$ and $x_1 = t+1$ have the same probability. Thus, by Lemma 1, $\Delta(t+1, x_2, x_3), C) = \Delta((t, x_2, x_3), C)$. Therefore, the maximization $\max_{s \in C_1} W(s, C_2)$ must have a solution that is an edge strategy.

Notice that given the constraints $2T < M < T^2/2$, the inequality $\Delta((x_1, x_2, x_3), C) \geq \frac{1}{2T}$ guarantees that increasing x_1 by one unit is strictly beneficial when playing against C, while the inequality $\Delta((x_1, x_2, x_3), C) \leq \frac{2}{T^2}$ implies that increasing x_1 by one unit is strictly harmful. ∎

CLAIM 1: *All cells besides* $(2, H)$ *and* $(3, mix)$ *are not MD-equilibria.*

PROOF:

$(1, H)$: Within $(1, H)$, the optimal strategy against the cell must be either $(T, 0, 0)$ or $(T/2+1, 0, 0)$, and they are inferior to strategies in $(2, H)$ and $(1, L)$, respectively:

$$
\begin{aligned}
u((T/2+1, T/2+1, T/2+1), (1, H)) &= M - \frac{3T}{2} - 3 \\
> u((T, 0, 0), (1, H)) &= M\left(\frac{7}{12} - \frac{1}{6T}\right) - T
\end{aligned}
$$

and

$$u((1,0,0),(1,H)) = \frac{5}{12}M - 1$$
$$> u((T/2+1,0,0),(1,H)) = \left(\frac{5}{12} + \frac{1}{6T}\right)M - \frac{T}{2} - 1.$$

$(1,L)$: The strategy $(1,1,1) \in (3,L)$ always wins against $(1,L)$, and its payoff of $M-3$ is higher than that of the two edge strategies in $(1,L)$: $u((1,0,0),(1,L)) = \left(\frac{5}{12} + \frac{1}{6T}\right)M - 1$ and $u((T/2,0,0),(1,L)) = (7/12)M - T/2$.

$(3,H)$: The strategy $(0,T,T) \in (2,H)$ is superior to $(T/2+1, T/2+1, T/2+1)$ because the expected payoffs of the former are approximately $M - 2T$ and of the latter $-\frac{3T}{2}$. Furthermore, $u((0,T,T),(3,H)) = \frac{T^3 - 2T^2 + T}{T^3}M - 2T > u((T,T,T),(3,H)) = \frac{T^3 - 3T + 2}{T^3}M - 3T$ since $T^2/2 > M$.

It is straightforward to verify that $\Delta((T/2+1, T/2+1, T),(3,H)) > \frac{1}{2T}$ (the strategy $(T/2+1, T/2+1, T)$ almost always loses in the second bid and wins in the third, and hence, the marginal expected gain is on the scale of $\frac{4}{T}\frac{1}{2}$), and thus $(T, T/2+1, T)$ is a better strategy against $(3,H)$.

The strategy $(T/2, T, T) \in (3, mix)$ is superior to $(T/2+1, T, T)$ against $(3,H)$ since $M\Delta((T/2,T,T),(3,H)) = M\left(\frac{T-1}{2T(T/2)^2}\right) < 1$.

$(3,L)$: Parallel arguments to those in the previous case for strategies consisting of bids in the category *High* apply to strategies consisting of bids in the category *Low*: $(1,1,1)$ and $(1,1,T/2)$ are not best responses to $(3,L)$, and the strategy $(0, T/2, T/2) \in (2,L)$ is superior to $(1, T/2, T/2)$. The other edge strategy $(T/2, T/2, T/2)$ is inferior to $(T/2+1, T/2+1, 1) \in (3, mix)$, which wins M with certainty and costs less.

$(2,L)$: As in step 1 of Claim 1, $(T/2, T/2, 0)$ is the best strategy within $(2,L)$ against the cell. The strategy $(T/2+1, T/2, 0) \in (2, mix)$ does better since $\Delta((T/2, T/2, 0),(2,L)) = \frac{1}{2T}$.

$(2, mix)$: We will show that for any $l \in Low$ and $h \in High$, the strategy $(1,l,h) \in (3, mix)$ is superior to $(0,l,h)$ against $(2, mix)$. The strategy $(1,l,h)$ increases the probability of winning M by at least $1/12$ (since it wins with probability 1 against any strategy in $(2, mix)$ of the form $(0, y_2, y_3)$, where $y_2 \in High$ and $y_3 \in Low$, whereas $(0,l,h)$ wins with probability $1/2$ against those strategies). Thus, its expected improvement is at least $M/12 - 1$, which is positive for $T \geq 6$. ∎

CLAIM 2: *The cell* $(2,H)$ *is an MD-equilibrium. It is not global unless* $\frac{3(T-2)}{T-1} > \frac{M}{T} > \frac{3T}{T-1}$.

PROOF:

Step 1: The strategy $(T,T,0)$ is an optimal strategy within $(2,H)$ against the cell itself and achieves an expected payoff of $\left(1-\frac{1}{T}\right)M-2T$. (Note that the payoff is positive whenever $M \geq 2T+3$).

For $(x_1,x_2,0) \in (2,H)$, we have $\Delta((x_1,x_2,0),(2,H)) = \Pr_{(2,H)}(y_1 \in \{x_1,x_1+1\}$ and $y_3 = 0)]/4 + \Pr_{(2,H)}(y_1 \in \{x_1+1,x_1\}$ and $y_2 = 0)/2$ $= \frac{1}{4}\frac{4}{T}\frac{1}{3} + \frac{1}{2}\frac{4}{T}\frac{1}{3} = \frac{1}{T}$. Since this marginal is greater than $\frac{1}{2T}$, the strategy $(T,T,0)$ is optimal within the cell against $(2,H)$.

The strategy wins against all strategies in the cell (which contains $3T^2/4$ strategies), with the exception of: (i) $T+1$ strategies of the type $(T,0,y_3)$, $(0,T,y_3)$, and $(T,T,0)$, which it wins against with probability $1/2$, and (ii) $T-2$ strategies in $(2,H)$ of the type $(T,y_2 < T,0)$ and $(y_1 < T,T,0)$, which it wins against with probability $3/4$. That is, it wins M with probability $1-\frac{(T+1)/2+(T-2)/4}{3\left(\frac{T}{2}\right)^2} = \left(1-\frac{1}{T}\right)$ and its expected payoff against $(2,H)$ is $\left(1-\frac{1}{T}\right)M-2T$.

Step 2: The cell $(2,H)$ is an MD-equilibrium.

We wish to verify that no strategy in $(1,H),(3,H),(2,mix)$, or $(2,L)$ achieves an expected payoff against $(2,H)$ larger than that of $(T,T,0)$.

$(1,H)$: For $(x_1,0,0) \in (1,H)$, we have $\Delta((x_1,0,0),(2,H)) = \Pr_{(2,H)}(y_1 \in \{x_1+1,x_1\}$ and $(y_2 = 0$ or $y_3 = 0)]/4 = \frac{1}{4}\frac{2}{3}\frac{4}{T} = \frac{2}{3T}$. Thus, the strategy $(T,0,0)$ is optimal in $(1,H)$ against $(2,H)$. The strategy wins M with probability $\frac{2}{3}\frac{1}{2}\left(1-\frac{1}{2}\frac{2}{T}\right) = \frac{T-1}{3T}$. Given the constraints, which imply also that $T \geq 6$, the strategy is inferior to $(T,T,0)$ $\left(\frac{T-1}{T}M-2T \geq \frac{T-1}{3T}M-T\right)$.

$(3,H)$: By Lemma 2, it is sufficient to show that no strategy in *edge*$(3,H)$ does better than $(0,T,T)$ against $(2,H)$.

Any strategy that involves a cost of at least $2T+T/2$ is inferior to $(0,T,T)$ since at most it adds $\frac{M}{T}-\frac{T}{2} < 0$ to the expected payoff.

The other edge strategies in $(3,H)$ are $(T/2+1,T/2+1,T)$, which wins only with probability of about $2/3$ but which costs the same as $(T,T,0)$, and $(T/2+1,T/2+1,T/2+1)$, which wins with probability $\frac{3\left(2\left(\frac{T}{2}-1\right)\frac{1}{2}+\frac{3}{4}\right)}{3\left(\frac{T}{2}\right)^2} \leq \frac{2}{T}$ and $M\frac{2}{T}-\frac{3T}{2}-3 < \left(1-\frac{1}{T}\right)M-2T$.

$(2,L)$: No strategy in $(2,L)$ wins M when playing against any strategy in $(2,H)$ and $\left(1-\frac{1}{T}\right)M-2T \geq -2$ given $M > 2T$.

$(2, mix)$: Placing a bid in *Low* that is higher than 1 is not optimal when playing against $(2, H)$. Therefore, an optimal strategy in $(2, mix)$ against $(2, H)$ is of the form $(x_1, 0, 1)$. Now, $\Delta((x_1, 0, 1), (2, H)) = \Pr_{(2,H)}(y_1 \in \{x_1 + 1, x_1\}$ and $y_2 = 0)]/4 + \Pr_{(2,H)}(y_1 \in \{x_1 + 1, x_1\}, y_3 = 0)/2 = \frac{1}{4}\frac{1}{3}\frac{4}{T} + \frac{1}{2}\frac{1}{3}\frac{4}{T} = \frac{1}{T}$ and thus $(0, 1, T)$ is an optimal strategy in $(2, mix)$ against $(2, H)$.

This strategy is inferior to $(0, T, T)$ since saving T does not justify the loss of M with probability of approximately $1/2$. Formally, $W((0, 1, T), (2, H)) = \frac{1}{2} - \frac{1}{2T}$ (i.e., it wins against any strategy in $(2, H)$ of the types $(0, y_2, T)$, $(0, y_2, y_3 < T)$, $(y_1, 0, y_3 < T)$, $(y_1, 0, T)$, and $(y_1, y_2, 0)$ with probabilities 1/4, 1/2, 1, 1/2, and 0, respectively) and $M\frac{T-1}{2T} - T - 1 \leq \frac{T-1}{T}M - 2T$ since $2T < M$.

Step 3: The MD-equilibrium $(2, H)$ is not global unless $\frac{3(T-2)}{T-1} > \frac{M}{T} > \frac{3T}{T-1}$.

The strategy $(1, 1, T) \in (3, mix)$ wins M with probability $2/3$ against $(2, H)$ and its expected payoff is about $(2/3)M - T$, which is greater than $M - 2T$ if $3T > M$. The strategy $(T, T, 1) \in (3, mix)$ costs one unit more than $(T, T, 0)$ and increases the probability of winning against any strategy of the form $(T, y_2, 0)$ or $(y_1, T, 0)$ by $1/4$. The frequency of such strategies in $(2, H)$ is approximately $\frac{1}{3}\frac{4}{T}$. Thus, the approximate expected improvement is $\frac{1}{4}\frac{4M}{3T} - 1$, which is positive if $M/T > 3$. Formally, if $\frac{3(T-2)}{T-1} > M/T$, then $u((1, 1, T), (2, H)) = \left(1 - \frac{1}{T}\right)M - T - 2 > \left(1 - \frac{1}{T}\right)M - 2T = u((0, T, T), (2, H))$. ∎

PROPOSITION 3: *In the edited tennis coach game with parameters T and c satisfying* $1/2 > c$ *and* $(3T - 1)c > 3/2$:

(*i*) *No cell* $(Q, strong)$ *is an MD-equilibrium.*

(*ii*) *If* $c \in (5/16, 3/8]$, *then the cell* $(3, not\ strong)$ *is a non-global MD-equilibrium for all T.*

PROOF OF PROPOSITION 3:

(i) The upper bound on the range of c guarantees the claim for $Q = 0, 1, 3T - 1$, and the lower bound guarantees the claim for $3T$.

Let $2 \leq Q \leq 3T - 2$. Note that the total score in any match between two teams with the same total skill level is either 1, 1.5, or 2.

Let $q = \min\{\lfloor Q/2 \rfloor, T\}$. We first show that either $a = (q, Q - 2q, q)$ or $b = (q, 0, q + 1)$, both of which are strategies in $(Q, not\ strong)$, scores 2 points against all strategies in $(Q, strong)$ except for two strategies against which they score 1.5 points.

For even Q or $3T - 2 \geq Q > 2T$, the strategy a scores 2 points against all strategies in $(Q, strong)$ that do not assign q to a side court (since none of the strategies in this cell assign more than q to a side court). The strategy a scores 1.5 points against the only two strategies in $(Q, strong)$ that assign q to one of the side courts, i.e., $(q, q, Q - 2q)$ and $(Q - 2q, q, q)$.

For odd $Q < 2T$, the strategy b always wins in the third court since there is no strategy $z \in (Q, strong)$ with $z_3 \geq q+1$. There are only two strategies in $(Q, strong)$ in which the tennis player in court 1 has a skill level q: $(q, q+1, 0)$ and $(q, q, 1)$, and b scores 1.5 points against these two strategies and 2 points against all others in $(Q, strong)$.

It remains to show that every $x \in (Q, strong)$ does worse than one of the strategies $a, b \in (Q, not\ strong)$ since it ties against at least two strategies in $(Q, strong)$ besides x. The following table specifies pairs of two strategies for every $x \in (Q, strong)$ for which $x_1 < x_3$ (the case of $x_3 > x_1$ is symmetric). Note that the strategies identified in the third and fifth rows in the table are in $(Q, not\ strong)$ because $Q \leq 3T-2$.

$x \in (Q, strong)$	Strategies in $(Q, not\ strong)$ that tie with x	
$x_1 = x_3 = 0$	$(1, x_2-1, 0)$	$(0, x_2-1, 1)$
$x_2 > x_1 = x_3 > 0$	(x_1-1, x_2, x_3+1)	(x_1+1, x_2, x_3-1)
$x_1 = x_2 = x_3$	(x_1-1, x_2+1, x_3)	(x_1, x_2+1, x_3-1)
$T > x_2 = x_1 > x_3$	(x_1-1, x_2, x_3+1)	(x_1-1, x_2+1, x_3)
$T = x_2 = x_1 > x_3$	(x_1-1, x_2, x_3+1)	(x_1-2, x_2, x_3+2)
$x_2 > x_1 > x_3 \geq 0$	(x_1-1, x_2, x_3+1)	(x_1, x_2-1, x_3+1)

(ii) Let $m_i(j) = u((j+1, x_{-i}), (3, not\ strong)) - u((j, x_{-i}), (3, not\ strong))$. Note that $m_i(j)$ is well-defined since it is independent of x_{-i}. Note also that $m_1(j) = m_3(j)$. Let $m_i = (m_i(0), m_i(1), \ldots)$ be the vector of marginals for the ith court.

For $T = 2$, the best strategy in $(3, not\ strong)$ is 210, which achieves an average score of 1.625. Any strategy in $(3, strong)$ scores only 1.5 points against the cell. It is straightforward to verify that (in this range of c) no strategy in $(Q, not\ strong)$ does better than 210 against $(3, not\ strong)$. The MD-equilibrium is not global in this range since the strategy $010 \in (1, strong)$ scores 1 point against all strategies in the cell and $1 - c > 1.625 - 3c$.

For $T \geq 3$, the cell $(3, not\ strong)$ is identical for every $T \geq 3$. The vectors of marginals are $m_2 = (6/12, 2/12, 0, 0, 0, \ldots)$ and $m_1 = m_3 = (3/12, 3/12, 3/12, 1/12, 0, \ldots)$, which are decreasing sequences. Thus, the best strategy for $Q = 3$ against the cell is 210, which achieves an average score of $10/6$. In order to verify that there is no better strategy in any cell $(Q, not\ strong)$, note that $c \geq 1/4$ guarantees that adding skill to the side courts is not beneficial. Furthermore, $c \leq 3/8$ guarantees that 100 is no better than 210 (since it reduces the average score by $3/4$, which is more than the saving of $2c$). Thus, $(3, not\ strong)$ is an MD-equilibrium. It is not global since the strategy 010 is superior to 210 against the equilibrium cell (since the saving of $2c$ is larger than the loss of $5/8$ on the first court). ∎

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