Micro Economics for Phd Q1: Exam Solution

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Question 1

Let M be the set of agents for which x^i is not a global maximum of i's preferences. For every $i \in M$, by differentiability of i's preferences, there is a vector v^i such that the set of i's improvement directions at x^i is $\{d|dv^i > 0\}$. Let $H^i = \{y|(y - x^i)v^i \leq 0\}$ (which is a half space). If there were a vector $y \in Y$ satisfying $(y - x^i)v^i > 0$ then by the convexity of Y there would be a point in Y on the interval between x^i and y that is strictly better than x^i. Thus, Y must be a subset of $\cap_{i\in M} H^i$. On the other hand, none of the vectors in H^i are strictly better for agent i than x^i and since x^i is optimal for agent i on H^i it is also optimal on the convex set $\cap_{i\in M} H^i$. Thus, Y must be the intersection of a number (not greater than the number of agents) of half-spaces.

b. Note that every agent's preference relation is single-peaked. There are two cases to consider: (i) some agents have peaks to the left of z and others to the right, or (ii) all agents' peaks are on one side of z (say, the right side).

(i) $Y = \{z\}$. The set Y is convex and achieves the dictator's goal. Any convex superset contains a point strictly between z and the peaks of agents either to the right or left of z. Thus, there is no convex set containing Y such that all agents choose z.

(ii) $Y = [0, z]$ is a convex set that achieves the dictator's goal. Any convex superset will include points to the right of z and to the left of the most leftward peak and thus will not achieve the dictator's goal.

Question 2

a. Attach to each candidate a number in $[0,1]$ interpreted as his position between $L=0$ and $R=1$. Assume that all candidates have distinct positions and identify a candidate according to his position. Let $N_A = \{0, 0.9\}$ and $N_B = \{0.1, 1\}$. Assume that the voter views two candidates close if the distance between their peaks is not more than 0.2. Then:

$C(\{0, 0.1\}) = 0$, $C(\{0.9, 0.1\}) = 0.1$, $C(\{0.9, 1\}) = 0.9$, and $C(\{0, 1\}) = 1$.

In order for a preference $\succsim$ to rationalize C, it must be that $0 \succ 0.1 \succ 0.9 \succ 1 \succ 0$, but the strict component of a preference relation does not have a cycle.

b. It is sufficient to show that $\rhd$ does not have cycles, since then it can be extended to a complete ordering and $C(\{x, y\}) = x$ iff $x \rhd y$.

Assume that there is a cycle. Since the voter's choice function is defined over doubletons of one candidate from A and one from B, no two candidates from the same party are related by $\rhd$ and any cycle must be of even length where members of A and B alternate. Consider a shortest cycle $a_1 \rhd b_1 \rhd a_2 \rhd b_2 \dot{\rhd} \cdots \rhd a_K \rhd b_K \rhd a_1$. Obviously, the relation $\rhd$ does not have a cycle of size 2 and by assumption it is not of length 4. Thus, it is at least 6 of length ($K \geq 3$). But, if $C(a_1, b_2) = a_1$ we can shorten the minimal cycle to $a_1 \rhd b_2 \dot{\rhd} \cdots \rhd a_K \rhd b_K \rhd a_1$ and if $C(a_1, b_2) = b_2$ we can shorten it to $a_1 \rhd b_1 \rhd a_2 \rhd b_2 \rhd a_1$, a contradiction.

c. If a voter's choice function fulfills the condition in part (b), then it can be rationalized by an ordering and since X is finite the ordering can be represented by some utility function u. Then, it is true that the choice function can be explained, as claimed, by attaching $v(x) = u(x) - b$ to every $x \in A$ and setting $v(x) = u(x)$ for every $x \in B$. But, it can also be explained analogously by attaching $v(x) = u(x)$ to every $x \in A$ and setting $v(x) = u(x) - b$ for every $x \in B$. In other words, the bonus is simply an arbitrary rescaling of the values of some candidates. The data does not make it possible to conclude that the voter has a positive tendency to either party.

Question 3a

C and not I: Let $\triangleright$ be an arbitrary strict ordering of the candidates. Recall that the positions are numbered $1, 2, \dots, K$. Given a profile, first assign the candidates to positions on which there is a consensus. Then, fill the remaining positions (starting from the smallest number and working up) according to $\triangleright$.

I and not C: Assign the same assignment to all profiles.

$K = 2$: Let $X = \{a, b\}$. Assigns a to position 1 unless all referees recommend that b be assigned to position 1.

Question 3b: Following is a proof for any n. The proof for $n = 2$ is much simpler.

Step 1: If x is assigned by the DR to position k, then at least one referee recommends that x is assigned to k.

Proof: Assume by contradiction that there exists a profile of recommendations $P1$ such that no referee recommends that x be assigned ro k and he is nonetheless assigned to k. Construct another profile $P2$ which coincides with $P1$ with regard to k and in which all referees recommend that x be assigned to some $l \neq k$. Regarding $P2$, by I, x is assigned to k and by C he is assigned to l, a contradiction.

Step 2: Definitions. Given a DR:

A group of referees M is *semi-decisive* for (k, x) (position k and candidate x) if *there is a profile* where the set of supporters of x being assigned to k is exactly M and indeed x is assigned to position k.

A group of referees M is *decisive* for (k, x) if whenever the set of supporters of x being assigned to k is exactly M then x is indeed assigned to k. It is *decisive* if it is decisive for all (k, x).

Step 3. If M is semi-decisive for (k^*, x^*) then M is decisive.

Proof: (i) $N - M$ is not semi-decisive for (l, x^*) for any $l \neq k^*$.

Denote by $P1$ a profile that qualifies M to be semi-decisive for (k^*, x^*). Assume by contradiction that there is a profile $P2$ for which the set of supporters for x^* to l is exactly $N - M$ and for which x^* is assigned to l. Form a profile $P3$ such that it is identical to $P2$ regarding l and to

$P1$ regarding k^* ($P3$ exists because $P1$ for k^* differs from $P2$ for l for all referees). By I, x^* is assigned in $P3$ to both k and l, a contradiction.

(ii) M is decisive for (k^*, x^*).

Assume by contradiction that M is not decisive for (k^*, x^*). Then there exists a profile $P1$ such that the set of supporters of x^* to k^* is exactly M and $y \neq x^*$ is assigned to k^*. Form a profile $P2$ identical to $P1$ regarding k^*, and for which all in $N - M$ recommend x^* to a certain $l \neq k^*$. By I, y is assigned in $P2$ to k^* and by (a) x^* (who is recommended by referees only to k^* and l) is assigned to l, violating the assumption that $N - M$ is not semi decisive for (l, x^*).

(iii) M is decisive for any (l, y) where $y \neq x^*$ and $l \neq k^*$.

Let $P1$ be a profile where M is the set of those recommend y for l. Let $P2$ be a profile such that regarding l it is identical to $P1$ and regarding k^* all M recommend x^* and all $N - M$ recommend y. By (ii) x^* is assigned to k^* in $P2$ and therefore by (a) y is assigned to l. By I y is assigned to l in $P1$. Thus, M is semi-decisive for (l, y), and by (ii) M is also decisive for (l, y). To prove that M is decisive for (l, x^*) or (k^*, y) apply the above twice.

Step 4. If M is decisive and $|M| > 1$, then M has a proper subset that is also decisive.

Proof: Let M be decisive and let $\{M_1, M_2\}$ be a proper partition of M. Let a, b, c be three candidates in X (here we use the assumption that $|X| \geq 3$). Take a profile in which regarding position 1 all of M_1 recommend b, all of M_2 recommend c and all other recommend a. Since M is decisive a it is not semi-decisive for $N - M$ and thus either, the DR assigns b to 1 and M_1 is semi-decisive for $(1, b)$ and thus M_1 is decisive, or c is assigned to position 1 and M_2 is semi-decisive for $(1, c)$ and thus is decisive.

Step 5: There is a referee i^* such that $\{i^*\}$ is decisive for all (k, x).

Proof: By C, the set of all referees is decisive. Let M be a minimal decisive set. By Step 5 it is a singleton.

Step 6: The referee i^* is a dictator.

Proof: Assume that there is a set $M \supseteq \{i^*\}$ which is not semi-decisive for some (k,x). Consider a profile where regarding position k all of M recommend x and all other recommend a certain $y \neq x$, and regarding a position $l \neq k$, i^* recommends y and all other $z \notin \{x,y\}$.

			k	l	
$i^* \in M$			x	y	
$\in M$			x	z	
$\in M$			x	z	
$\notin M$			y	z	
$\notin M$			y	z	

Since M is not semi-decisive for (k,x) and by step 1, the DR assigns y to k. The set $\{i^*\}$ is decisive and thus y is assigned also to l, a contradiction.