Exam in Microeconomics for Phd.
NYU Economics

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Date: October 21st, 2021
Time: 09:00 - 12:00
Instructions: You are asked to answer the three questions. It is an exam with "open-books" and you can use any written resources. Obviously you are forbidden from communicating with anybody during the exam.

Question 1. Let A be a set of at least three objects. A distance function d assigns a number in $[0,1]$ to every pair of objects such that for every $x, y, z \in A$

$d(x,x) = 0$; $d(x,y) = d(y,x)$; and $d(x,y) + d(y,z) \geq d(x,z)$.

Let $N = \{1,..,n\}$ be a set of individuals. Each individual i holds a distance function on A.

An aggregator F assigns a distance function to every profile of distance functions $(d_1,..,d_n)$.

An aggregator is *simple* if there exists a function f which assigns a number in $[0,1]$ to every vector of distances $(\alpha_1,...,\alpha_n)$ such that:

(i) f generates F in the sense that $F(d_1,..,d_n)(x,y) = f(d_1(x,y),\ldots,d_n(x,y))$.

(ii) *Unanimity:* $f(c,\ldots,c) = c$ for all $c \in [0,1]$

(iii) *Anonymity:* $f(\alpha_1,\ldots,\alpha_n) = f(\beta_1,\ldots,\beta_n)$ if $(\beta_1,\ldots,\beta_n)$ is a permutation of $(\alpha_1,\ldots,\alpha_n)$.

(a) Assume that the distance between any two objects must be either 0 or 1. Find a simple aggregator.

(b) Assume that the distance between two objects can be any number in $[0,1]$. Find another example of a simple aggregator.

(c) Show that there is only one correct answer to (a).

Question 2. Professors of economics evaluate job candidates. The set of candidates is large but eventually the choice will be made from a subset containing an odd number of candidates. Professors care only about the candidate's attitude towards behavioral economics, which is measured by a non-zero real number (for example, $+3$ is more positive than $+1$ and -50 is negative). Let $X = \mathit{Reals} \setminus \{0\}$. Assume that no two candidates have the same attitude. Thus, each professor can be described as a choice function that selects one number from each odd cardinality set of numbers.

Both professors wish to select a candidate who best reflects the mood of the profession. Both believe that the attitude to behavioral economics is distributed according to a Normal distribution: either $N(+2, 1)$ or $N(-2, 1)$.

When facing a set of candidates Y:

Professor A finds the distribution that best explains Y in the sense of maximum likelihood (that is, $N(+2, 1)$ if $\sum_{x \in Y} x > 0$, and $N(-2, 1)$ if $\sum_{x \in Y} x < 0$) and then chooses the candidate in Y who is the closest to the peak of that distribution.

Professor B divides Y into two groups: positive candidates (closer to $+2$) and negative candidates (closer to -2). He chooses the candidate in Y with a positive attitude who is closest to $+2$ if the majority of candidates are positive, and the candidate with a negative attitude who is closest to -2 if the majority are negative.

(0) Formalize the two professors as choice functions c_A and c_B.

(1) Is either professor rationalizable as a maximizer of a preference relation on X (the set of possible attitudes toward b.e.)? Prove your answer.

(2) Show that the following property of choice functions is satisfied by the two professors: For any $X \supseteq Y_1 \supset Y_2 \supset Y_3$, if $a = c(Y_1) \neq c(Y_2) = b$ and $a, b \in Y_3$, then $c(Y_3) \in \{a, b\}$.

(3) Show that the following property distinguishes between the two professors: If $c(Y) = c(Z) = a$ and $Y \cap Z = \{a\}$, then $c(Y \cup Z) = a$ (that is, the property is satisfied by one professor but not the other).

Question 3. A *test* is a finite vector of 0's and 1's. Let X be the set of tests. The interpretation of the test $(1, 0, 1, 1)$ (for example) is that a student will be asked to answer one of four questions; he fails the test if asked the second and passes if he asked one of the other three. Define $n(s)$ to be the number of possible questions in test s (i.e. the length of the vector s).

A compound test $s \oplus t$ (where $s, t \in X$) is a test in which the student will be given one of the two tests s or t. The student identifies the compound test as a test (vector) of length $n(s_1) + n(s_2)$ where the vector s is first and the second is t. Thus, for example, if $s = (1, 1, 1)$ and $t = (1, 0)$ then $s \oplus t = (1, 1, 1, 1, 0)$.

Let $\succsim$ be a preference relation over X satisfying the following properties:

Symmetry: if s is a permutation of t then $s \sim t$.

Monotonicity: any sequence of ones is better than any sequence of zeroes.

Independence (I): if $s \succsim t$ if and only if $s \oplus r \succsim t \oplus r$ for every tests r, s, t.

(a) Interpret I. Explain why $\succsim$ cannot be the preference relation with the utility representation $u(s) =$"the proportion of ones in s".

(b) Given one example of the many preference relations that satisfy the three properties.

(c) Show that it is impossible that both $(0, 0) \sim (0)$ and $(1, 1) \sim (1)$.

(d) Show that there is only one preference relation satisfying the three assumptions and $(1, 1) \sim (1)$.

Question 1 – Solution:

(a) Define the aggregator F by $F[d_1, \ldots, d_N](x, y) = \max\{d_i(x, y)\}$.

F is a distance function since:

(1) $\max\{d_i(x, x)\} = 0$ for all $x \in A$;

(2) $\max\{d_i(x, y)\} = \max\{d_i(y, x)\}$ for all $x, y \in A$; and

(3) if $d_i(x, z) = \max\{d_i(x, z)\}$ then for any $x, y, z \in A$:

$F[d_1, \ldots, d_n](x, z) = d_i(x, z) \le d_i(x, y) + d_i(y, z) \le F[d_1, \ldots, d_n](x, y) + F[d_1, .., d_n](y, z)$

Obviously, F is generated by $f(\alpha_1, \ldots, \alpha_N) = \max\{\alpha_i\}$ which satisfies Unanimity and Anonymity.

(b) Let $F[d_1, d_2, \ldots, d_n](x, y) = \frac{1}{n}\sum_{i=1}^{n} d_i(x, y)$.

For every profile of distance functions $(d_1, .., d_n)$, $F[d_1, d_2, \ldots, d_n]$ is the distance function since:

(1) $d_i(x, x) = 0$ for all $x \in A$ and distance functions d_i;

(2) $d_i(x, y) = d_i(y, x)$ for all $x, y \in A$ and distance functions d_i; and

(3) for all $x, y, z \in A$,

$\frac{1}{n}\sum_{i=1}^{n} d_i(x, z) \le \frac{1}{n}\sum_{i=1}^{n} (d_i(x, y) + d_i(y, z)) = F[d_1, \ldots, d_n](x, y) + F[d_1, \ldots, d_n](y, z)$

The aggregator F is generated by $f(\alpha_1, .., \alpha_n) = \frac{1}{n}\sum_{i=1}^{n} \alpha_i$ which satisfies Unanimity and Anonymity.

(c) Let F be a simple aggregator generated by f. Let k^* be the maximal k for which there is a vector $(\alpha_1, \ldots, \alpha_n)$ with k ones, such that $f(\alpha_1, \ldots, \alpha_n) = 0$. Assume by contradiction that $k^* > 0$. Since f satisfies Unanimity, $k^* < n$.

Let a, b, c be three elements in A.

Consider the profile of distance functions described by the following table:

agent i	$d_i(a, b)$	$d_i(b, c)$	$d_i(a, c)$	other $d_i(x, y)$ $(x \neq y)$
1	0	1	1	1
2	1	0	1	1
3	1	1	1	1
$\vdots$	1	1	1	1
$k^* + 1$	1	1	1	1
$k^* + 2$	0	0	0	1
$\vdots$	0	0	0	1
n	0	0	0	1

By the definition of k^*, the distance function $F(d_{.1}, \ldots, d_n)$ must assign 0 to (a, b) and to (b, c) and 1 to (a, c), contradicting the triangle inequality.

Question 2 – Solution:

(0) Professor A, when facing the set Y, calculates the difference between the loglikelihoods of Y given $N(2,1)$ and $N(-2,1)$, which is given by:

$\sum_{x\in Y}\frac{(x-2)^2}{2}-\sum_{x\in Y}\frac{(x+2)^2}{2}=2\sum_{x\in Y}x$. Thus,

$$C_A(Y)=\begin{cases} argmin_{x\in Y}|x-2| & \sum_{x\in S}x>0 \\ argmin_{x\in Y}|x+2| & \sum_{x\in S}x<0 \end{cases}$$

Professor B's choice depends on whether the majority of candidates are positive or negative. In the former case, he chooses the positive candidate closest to 2; in the latter case, he chooses the negative candidate closest to -2:

$$C_B(Y)=\begin{cases} argmin_{x\in Y\cap\mathbb{R}_+}|x-2| & |\{x\in Y|x>0\}|>|\{x\in Y|x<0\} \\ argmin_{x\in Y\cap\mathbb{R}_-}|x+2| & \text{otherwise} \end{cases}$$

(1) Neither professor satisfies condition α and therefore neither is rationalizeable. For example, let $Y=\{-2,1,2\}$ and $Z=\{-3,-2,-1,1,2\}$. Then, $C_A(Z)=C_B(Z)=-2$, while $C_A(Y)=C_B(Y)=2$ although $-2\in Y$, violating α for each of the choice functions.

(2) Assume $Y_1\supset Y_2\supset Y_3$, $a=c(Y_1)\neq c(Y_2)=b$ and $a,b\in Y_3$.

Professor A: Since a and b are in all three sets, it must be that one of them is closer to $+2$ than the other members of Y_3 and the other is closest to -2 out of all the members of Y_3; thus, one of them is $C_A(Y_3)$.

Professor B: It must be that a and b do not have the same sign, i.e. one is positive and the other negative. One of them will be the closest to $+2$ among the positive members of Y_3 and the other will be the closest to -2 among the negative members of Y_3 and therefore $C_B(Y_3)\in\{a,b\}$.

(3) Professor A doesn't satisfy this property. Let $Y = \{4, -1, -2\}$ and $Z = \{4, -3, -0.5\}$. Then, $C_A(Y) = C_A(Z) = 4$, but $C_A(Y \cup \{-3, -0.5\}) = -2$.

Professor B does satisfies this property: Assume $Y \cap Z = \{a\}$ and $C_B(Y) = C_B(Z) = a$. WLOG, $a > 0$. Then, a is closest to $+2$ in both Y and Z. Furthermore, a majority of members in both Y and Z are positive. Thus, a majority of elements in $Y \cup Z$ are also positive, and a is the closest member to $+2$ in $Y \cup Z$; that is, $C_B(Y \cup Z) = a$.

Question 3 – Solution :

a) (I) means that an agent compares two compound tests with some common questions by comparing the questions that differ between them – i.e. they 'cancel' out questions that are present in both before making a comparison.

Consider the preferences represented by u. Let $s = (1, 0)$, $t = (1, 1, 0, 0)$ and $r = (1)$. $u(s) = u(t)$ but $u(s \oplus r) > u(t \oplus r)$. b) Ranking tests by the

number of 1's (the more the better).

c) Suppose $(1, 1) \sim (1) \succ (0, 0) \sim (0)$. Then, by I, $(1, 0) \sim (1, 1, 0)$, $(1, 0, 0) \sim (1, 0)$, and $(1, 1, 0) \succ (0, 1, 0) \sim (1, 0, 0)$, and therefore $(1, 0) \sim (1, 1, 0) \succ (1, 0, 0) \sim (1, 0)$, a contradiction.

d) Suppose $(1) \sim (1, 1)$. Then, by I $(1, 0) \sim (1, 1, 0)$. Since $(1) \succ (0)$, then by I we also have $(1, 0, 1) \succ (1, 0, 0)$. Together, we have $(1, 0) \sim (1, 1, 0) \sim (1, 0, 1) \succ (1, 0, 0)$ and therefore by I $(0) \succ (0, 0)$.

Applying I, any sequence of k ones is better than a sequence of l zeros if $k < l$ and is indifferent between all sequences of constant 1.

Let $m[0] \oplus n[1]$ be the sequence of m zeros and n ones. By symmetry for every $x \in X$ there are integers x_0, x_1 such that $x \sim x_0[0] \oplus x_1[1]$.

Applying I multiple times we get:

$x \sim x_0[0] \oplus x_1[1] \sim x_0[0] \oplus 1[1] \succsim y_0[0] \oplus 1[1] \sim y_0[0] \oplus y_1[1] \sim y$

iff $x_0 \leq y_0$.

This result implies that if he is indifferent between tests that he will pass for sure, then he prefers a test with the least number of possibilities.