Ariel Rubinstein: **Micro-Theory**: **NYU**
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Exam: **SOLUTION**

Q1

Consider an economic agent with preferences $\succsim^1$ on the set of the bundles in a K-commodity world. The agent holds a bundle x^* and can consume any part of it; however, he feels obliged to give to his friend (who holds the preference relation $\succsim^2$) a bundle which will be at least as good for his friend as a fixed bundle y^*. Assume that $x_k^* > y_k^*$ for all k. Both preference relations satisfy strong monotonicity, continuity and strict convexity.

(1) State the agent's problem and explain why a solution exists and is unique.

(2) Denote the bundle the agent consumes given x^* as $z(x^*)$. The agent's indirect preferences on the space of initial bundles can be defined by $a^* \succsim^* b^*$ if $z(a^*) \succsim^1 z(b^*)$. Show that the indirect preferences are strictly convex and continuous.

(3) Show that if $\succsim^1$ is differentiable then so is $\succsim^*$.

Solution

(1) Agent 1 seeks a $\succsim^1$ -maximal bundle x from $B(x^*) = \{x \mid x^* - x \succsim^2 y^*\}$. By the continuity of $\succsim^2$, the set $B(x^*)$ is closed and clearly bounded. Thus, since $\succsim^1$ is continuous, a solution to the problem exists. The set $B(x^*)$ is also convex: if $a, b \in B(x^*)$, then $x^* - a \succsim^2 y^*$ and $x^* - b \succsim^2 y^*$ and by the the convexity of $\succsim^2$, $\lambda(x^* - a) + (1 - \lambda)(x^* - b) = x^* - (\lambda a + (1 - \lambda)b) \succsim^2 y^*$ and $\lambda a + (1 - \lambda)b \in B(x^*)$. By the strict convexity of $\succsim^1$, there cannot be two solutions to the agent's problem.

(2) Continuity: By the maximum theorem, the function z is continuous. Suppose that $a_n^* \succsim^* b_n^*$, $a_n^* \to a^*$ and $b_n^* \to b^*$. Then, $z(a_n^*) \succsim^1 z(b_n^*)$, and therefore by the function z's continuity we have $z(a^*) \succsim^1 z(b^*)$ and thus $a^* \succsim^* b^*$.

Convexity: Suppose that $a \succsim^* b$, that is $z(a) \succsim^1 z(b)$. Consider $\lambda a + (1 - \lambda)b$, where $\lambda \in (0, 1)$. The bundle $\lambda z(a) + (1 - \lambda)z(b) \in B(\lambda a + (1 - \lambda)b)$ since

$\lambda a + (1 - \lambda)b - (\lambda z(a) + (1 - \lambda)z(b)) = \lambda(a - z(a)) + (1 - \lambda)(b - z(b)) \succsim^2 y^*$, which is because both $a - z(a) \succsim^2 y^*$ and $b - z(b) \succsim^2 y^*$. Thus,

$z(\lambda a + (1-\lambda)b) \succsim^1 \lambda z(a) + (1-\lambda)z(b) \succ^1 z(b)$. It follows that $\lambda a + (1-\lambda)b \succ^* b$.

(3) By the differentiability of $\succsim^1$, there exists a vector v^* such that $d \cdot v^* > 0$ iff d is an improving direction at $z(x^*)$. We will show that v^* is a vector of "local values" for the relation $\succsim^*$ as well.

(i) Assume $z(x^*) + d \succ^1 z(x^*)$. Then $z(x^* + d) \succsim^1 z(x^*) + d \succ^1 z(x^*)$ and thus $x^* + d \succ^* x^*$. Therefore, any improving direction of $\succsim^1$ is also an improving direction for $\succsim^*$.

(ii) Suppose that d is an improving direction of $\succ^*$ and $dv \leq 0$. We can assume that $x^* + d \succ^* x^*$ and by continuity we can assume that $dv < 0$. By (i), $x^* - d \succ^* x^*$. The two "inequalities" contradict the strict convexity of $\succ^*$.

Q2

A decision maker who compares vectors (x_1,x_2) and (y_1,y_2) in R^2_+ is implementing the following procedure, denoted by $P(v_1,v_2)$, where for $i = 1,2$, v_i is a strictly increasing continuous function from the nonnegative numbers to the real numbers satisfying $v_i(0) = 0$:

(1) if one of the vectors dominates the other he evaluates it being superior.

(2) if $x_1 > y_1$ and $y_2 > x_2$, he carries out a "cancellation" operation and then makes the evaluation by comparing $(x_1 - y_1, 0)$ to $(0, y_2 - x_2)$, which is accoplished by comparing $v_1(x_1 - y_1)$ with $v_2(y_2 - x_2)$ (similarly, if $x_1 < y_1$ and $x_2 > y_2$, he bases his preference on the comparison of $v_2(x_2 - y_2)$ to $v_1(y_1 - x_1)$).

(a) Verify that if $v_i^*(t) = t$ (for both i), then the procedure $P(v_1^*, v_2^*)$ induces a preference relation on R^2_+.

(b) Explain why $P(v_1,v_2)$ does not necessarily lead to a transitive preference relation.

(c) Complete and prove the following proposition: If the procedure $P(v_1,v_2)$ induces a preference relation, then that preference relation is represented by....

Solution

(a) The preference relation represented by $x_1 + x_2$ is the relation induced by the procedure.

(b) Let $v_1(\Delta) = \Delta$ and $v_2(\Delta) = \Delta^2$. Consider $x = (2,0)$, $y = (0,2)$ and $z = (1,1)$. Then $y \succ x$ since $(2-0)^2 > (2-0)$, $z \sim y$ and $z \sim x$.

(c) Claim: If $P(v_1,v_2)$ induces a preference relation, then it can be represented by a utility function of the form $\alpha_1 x_1 + \alpha_2 x_2$.

The claim follows from the result in Lecture 4 which states that any preference relation on R^2_+ satisfying the following three properties can be represented by a utility function of the form $\alpha_1 x_1 + \alpha_2 x_2$:

(i) Quasi-linearity in both dimensions: If $x \succsim y$ and $v_1(x_1 - y_1) > v_2(y_2 - x_2)$, then also $v_1((x_1 + \epsilon e_{i1}) - (y_1 + \epsilon e_{i1})) \; > v_2((y_2 + \epsilon e_{i2}) - (x_2 + \epsilon e_{i2}))$, so that $x + \epsilon e_i \succsim y + \epsilon e_i$. Furthermore, if x dominates y, then adding ϵ units of any component preserves dominance.

(ii) Continuity: If $x \succ y$, then $v_1(x_1 - y_1) > v_2(y_2 - x_2)$ and by the continuity of v_1 and v_2 there exist neighborhoods of x and y such that the inequality still holds for all pairs in the two neighborhoods. If x dominates y then proving the continuty is trivial.

(iii) Strict monotonicity.

Question 3

Discuss the attitude of an agent towards lotteries over a set of consequences $Z = \{a, b, c\}$ satisfying that he ranks a first and c last.

Consider any preference relation (on $L(Z)$) satisfying independence and continuity. Obviously, each preference relation can be described by a single number $v \in (0,1)$ by attaching the numbers $1, v, 0$ to the three alternatives. Denote this preference relation by $\succsim_v$.

For a set $V \subseteq (0,1)$, define a choice correspondence $C_V(A)$ as the set of all $p \in A$ satisfying that there is no $q \in A$ such that $q \succ_v p$ for all $v \in V$.

Define the binary relation pD^*q if $p(a) \geq q(a)$ and $p(a) + p(b) \geq q(a) + q(b)$ with at least one strict inequality. Consider the choice correspondence C defined by $p \in C(A)$ if there is no $q \in A$ such that qD^*p. Show that $C = C_V$ for some set V.

Solution:

First note that $E_v(p) = p(a) + vp(b) = p(a)(1-v) + v(p(a) + p(b))$.

Let $V = (0,1)$. We will show that $C(A) = C_V(A)$ for all A.

Suppose that $p \in C(A)$. Then, for no $q \in A - \{p\}$ we have

$$\begin{cases} q(a) \geq p(a) \\ q(a) + q(b) \geq p(a) + p(b) \end{cases}$$

(the condition that $q \neq p$ is equivalent to "with at least one strict inequality".)

If there were q such that $q \succ_v p$ for all v, then $q(a) + vq(b) > p(a) + vp(b)$ for all $v \in (0,1)$, which implies that both:

- $q(a) \geq p(a)$ (take the limit of $v \to 0$)
- $q(a) + q(b) \geq p(a) + p(b)$ (take the limit of $v \to 1$).

Thus, $p \in C_V(A)$.

Suppose that $p \notin C(A)$. Then, there exists q such that $q(a) \geq p(a)$ and $q(a) + q(b) \geq p(a) + p(b)$, with at least one strict inequality. Then,

$E_v(q) = q(a)(1-v) + v(q(a) + q(b)) \; > E_v(p) = p(a)(1-v) + v(p(a) + p(b))$ for all $v \in (0,1)$, and therefore $p \notin C_V(A)$.