A draft of "solution":

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Problem 1

1. Consider a decision maker on the space $X = [0,1]$ where $t \in X$ is interpreted as the portion of the day he contributes to society.

(1) Assume that he has a strictly convex and continuous preference relation over X. Show that he has a "single peak" preference relation, namely that there exists x^* such that for every $x^* \leq y < z$ or $z < y \leq x^*$ he strictly prefers y to z. Find a strictly convex preference relation on this space which is not continuous.

(2) Assume that the domain of the decision maker's choice function contains all sets of the form $B(w,\rightarrow) \; = \{x \in X \mid x \geq w\}$, as well as of the form $B(w,\leftarrow) \; = \{x \in X \mid x \leq w\}$, where $w \in [0,1]$. Interpret these sets. Show that the decision maker's choice function induced from a strictly convex and continuous preference relation is always well-defined and continuous in w.

Solution to Problem 1

(1) Since X is compact and $\succeq$ is continuous, there exists a $\succeq$-maximal element in X; it is also unique, since $\succeq$ is strictly convex and X is a convex set.

Let x^* be the unique $\succeq$-maximal element in X. Consider any $z < y \leq x^*$. If $y = x^*$, then by definition of x^* we have $y \succ z$. If $y < x^*$, we assume by contradiction that $z \succeq y$. Since also $x^* \succeq y$, by strictly convexity any element strictly between x and y is strictly better than y, which implies $y \succ y$, a contradiction. So it must be that $y \succ z$. Similarly, we can show that for every $x^* \leq y < z$, $y \succ z$.

Among preference relations that are convex but not continuous, consider the one represented by the following utility function:

$$u(x) = \begin{cases} 0 & \text{if } x \leq 0.5; \\ 0.5 - x, & \text{if } 0.5 < x \leq 1. \end{cases}$$

(2) These are the sets that require the agent to contribute at most/least a proportion w of the day.

Note first that for all $w \in [0,1], a \in \{\rightarrow, \leftarrow\}$, there exists a $\succeq$-maximal element on $B(w,a)$ because $\succeq$ is continuous and $B(w,a)$ is compact; it is also unique, because $\succeq$ is strictly convex and $B(w,a)$ is a convex set. We can calculate $C_{\succeq}(B(w,\rightarrow)$ and $C_{\succeq}(B(w,\leftarrow)$ as:

$$C_{\succeq}(B(w,\rightarrow)) = \begin{cases} x^*, & \text{if } w \leq x^*; \\ w, & \text{if } w > x^*. \end{cases}$$

$$C_{\succeq}(B(w,\leftarrow)) = \begin{cases} w, & \text{if } w \leq x^*; \\ x^*, & \text{if } w > x^*. \end{cases}$$

These functions are clearly continuous in w.

Problem 2

Consider two types of decision makers:

Type A has in mind several criteria $(\succ_i)_{i \in I}$ where each $\succ_i$ is an ordering of the elements in a finite set X. Whenever the agent chooses from a set $A \subseteq X$ he is satisfied with any element a such that for any other $b \in A$ there is some i (i probably depends on b) for which $a \succ_i b$.

Thus, for example, if he has one criterion in mind then the induced choice correspondence picks the unique maximal element from each set ; if he has two in mind, where one is the negation of the other, then the induced choice correspondence is $C(A) \equiv A$.

(1) Show that if $a \in C(A) \cap C(B)$, then $a \in C(A \cup B)$.

(2) Suggest another interesting property that the choice correspondence induced by the above procedure always satisfies.

Type B has in mind a transitive asymmetric relation $\succ$ with the interpretation that if $a \succ b$ then he will not choose b if a is available. He is described by the choice correspondence $C(A) = \{x \in A|$ there is no $y \in A$ such that $y \succ x\}$.

(3) Show that every type A agent can be described as a type B agent.

(4) Show that every type B agent can be described as a type A agent.

Solution to Problem 2

(1) Since $a \in C(A)$, we have $\forall b \in A, b \neq a, \exists i, a \succ_i b$. Also, since $a \in C(B)$, we have $\forall c \in B, c \neq a, \exists i, a \succ_i c$. But this directly implies that $\forall d \in A \cup B, d \neq a, \exists i, a \succ_i d$. Then by definition, $a \in C(A \cup B)$.

(2) Following are two posisble properties:

Condition α. Consider any $a \in A \subset B$. Suppose that $a \in C(B)$. We want to show that $a \in C(A)$. Note that since $a \in C(B)$, we have $\forall b \in B, b \neq a, \exists i, a \succ_i b$. Since $A \subset B$, this directly implies that $\forall b \in A, b \neq a, \exists i, a \succ_i b$, and therefore $a \in C(A)$.

Path independence. $C(A) = C(C(A_1) \cup C(A_2))$, **for any** $A \in X$ **and any** $\{A_1, A_2\}$ **that is a partition of** A. Consider any A and any of its partitions $\{A_1, A_2\}$ and consider $a \in C(A)$. WLOG $a \in A_1$. By condition α, we know that $a \in C(A_1)$ and therefore $a \in C(A_1) \cup C(A_2)$. Since $C(A_1) \cup C(A_2) \subset A$, again by condition α we have $a \in C(C(A_1) \cup C(A_2))$. Therefore, $C(A) \subseteq C(C(A_1) \cup C(A_2))$.

Conversely, consider any $x \notin C(A)$. Then $\exists y$, s.t. $y \succ_i x, \forall i$. Note that for any partition such that x, y are in the same set, say $x, y \in A_1$, it must be that $x \notin C(A_1)$, because y dominates x in A_1 according to all criteria. Now consider some partition of A s.t. x, y are not in the same set, say $x \in A_1, y \in A_2$. In this case, if $y \in C(A_2)$, then even if $x \in C(A_1)$, we know that $x \notin C(C(A_1) \cup C(A_2))$. If $y \notin C(A_2)$, then $\exists z \in C(A_2)$ s.t. $z \succ_i y \succ_i x, \forall i$. (Note that $\exists z_1 \in A_2$ s.t. $z_1 \succ_i y, \forall i$. If $z_1 \notin C(A_2)$, then $\exists z_2$ s.t. $z_2 \succ_i z_1 \succ_i y, \forall i$. we continue until we find a $z \in C(A_2)$ s.t. $z \succ_i y, \forall i$.) Then, again even if $x \in C(A_1)$, we know that $x \notin C(C(A_1) \cup C(A_2))$. Therefore, in any case, $x \notin C(C(A_1) \cup C(A_2))$.

(3) Define $x \succ y$ iff $x \succ_i y, \forall i$. We denote by C_A the choice correspondence according to $\{\succ_i\}$ and by C_B the choice correspondence according to $\succ$. That is, $C_A(A) = \{a \in A | \forall b \in A, b \neq a, \exists i, a \succ_i b\}$ and $C_B(A) = \{a \in A | \nexists b \in A, b \succ a\}$. We want to show that $C_A(A) = C_B(A), \forall A \in X$.

Consider $a \in C_A(A)$. By the definition of C_A, $\forall b \in A, b \neq a, \exists i, a \succ_i b$. By the definition of $\succ$, we do not have $b \succ a, \forall b \in A$. Then, by definition of C_B, $a \in C_B(A)$.

Conversely, take any $a \in C_B(A)$. By definition of C_B, $\nexists b \in A, b \succ a$. However, in that case, the definition of $\succ$, it must be that $\forall b \in A, b \neq a, \exists i, a \succ_i b$; otherwise there would be some $b \in A, b \succ_i a, \forall i$, which implies $b \succ a$, a contradiction. However, this directly corresponds to the definition of C_A and implies that $a \in C_A(A)$.

(4) For every type B agent, define $\{\succ_i\}$ to be the collection of complete orderings on X that is extended from $\succ$. By Problem 4 in PS1 we know that $\{\succ_i\}$ is not empty, and by Problem 5 in PS1, we know that $\{\succ_i\}$ satisfies: $\forall x, y \in X, x \succ y$ iff $x \succ_i y, \forall i$. By what we have shown in (3), the choice correspondence defined by $\succ$ (type B agent) is exactly the same as the one defined by $\{\succ_i\}$ (type A agent), which is what needs to be shown.

Problem 3

Define an "amount of money" to be any positive integer. Define a "wallet" to be a collection of amounts of money. Denote the wallet with K amounts of money $x_1, .., x_K$ by $[x_1, .., x_K]$. Thus, for example, the wallet [3,3,4] with a total of 10 is identical to the wallet [4,3,3] and is different than the wallet [3,4] which has a total of 7. Let X be the set of all wallets. The following are two properties of preference relations over X :

Monotonicity:

(i) Adding an ammount of money to the wallet or increasing one of the amounts is weakly improving.

(ii) Increasing all amounts is strictly improving.

Split aversion:

Combining two amounts of money is (at least weakly) improving (thus [7,3] is at least as good as [4,3,3]).

(1) Let v be a function defined on the natural numbers satisfying (i) $v(0) = 0$, (ii) it is strictly increasing and (iii) superadditivity ($v(x+y) \geq v(x) + v(y)$ for all x, y). Show that the function $u([x_1, \ldots, x_K]) = \Sigma_{k=1,,.,K} v(x_k)$ is a utility function which represents a preference relation on X that satisifies monotonicity and split aversion.

(2) Give an example of a preference relation satisfying monotinicity but not split aversion and an example of a preference relation satsifying split aversion but not

monotonicity.

(3) Define the notion that one preference relation is more split averse than the other.

(4) Find a preference relation (satisfying monotonicity and split aversion) which is less split averse than any other split averse and monotonic preference relation.

(5) Show that the relation represented by the function $u([x_1,\ldots,x_K]) = \max\{x_1,..,x_K\}$ is more split averse than any preference relation of the type described in part (1).

Solution to Problem 3

(1) Monotonicity: Consider the wallets $[x_1,\ldots,x_K]$ and $[x_1,\ldots,x_K,x_{K+1}]$. We have $u([x_1,\ldots,x_K,x_{K+1}]) = u([x_1,\ldots,x_K]) + v(x_{K+1}) \geq u([x_1,\ldots,x_K])$.

Take any wallets $[x_1,\ldots,x_K]$ and $[y_1,\ldots,y_K]$ s.t. $y_i \geq x_i, \forall i$ with strict inequality for some i. Then $u([x_1,\ldots,x_K]) = \sum_{k=1}^{K} v(y_k) > \sum_{k=1}^{K} v(x_k)$, because v is strictly increasing and $y_i \geq x_i, \forall i$ with strict inequality for some i.

Split aversion: Consider the wallets $[x_1,\ldots,x_K]$ and $[x_1,..,x_{i-1},y_i,y_i',x_{i+1},..,x_K]$ where $x_i = y_i + y_i'$. We have:

$$u([x_1,\ldots,x_K]) - u([x_1,..,x_{i-1},y_i,y_i',x_{i+1},..,x_K]) = v(x_i) - (v(y_i) + v(y_i')) \geq 0$$

where the inequality follows from the superadditivity of v.

(2) 1. Monotonic but not split averse: $u([x_1,\ldots,x_K]) = \min\{x_1,\ldots,x_K\}$.

2. Split averse but not monotonic: $u([x_1,\ldots,x_K]) = -K$ (the more amounts there are the worst off one is).

(3) We say $\succeq_1$ is *more split averse than* $\succeq_2$ if for any wallets x and an amount c, $x \succeq_1 (c)$ implies $x \succeq_2 (c)$.

(4) Consider the preference relation $\succeq$ on X represented by $u([x_1,\ldots,x_K]) = \sum_{k=1}^{K} x_k$. We claim that $\succeq$ is less split averse than any other monotonic and split averse preference relation on X.

To see this, take any $\succeq'$ on X that is monotonic and split averse. Take any $x \in X$ and a number c s.t. the cardinality of y is 1. We want to show that $x \succeq' (c)$ implies $x \succeq (c)$. Since $\succeq'$ is split averse (applied inductively), then $(\sum_k x_k) \succeq' x$, by transitivity

$(\sum_k x_k) \succeq' (c)$ and by the monotonicity of $\succeq'$ we have $\sum_k x_k \geq c$ which implies $x \succeq c$.

(5) Let $\succeq$ be the preference relation represented by $u([x_1, \ldots, x_K]) = \max\{x_1, \ldots, x_K\}$. Let $\succeq'$ be any preference relation described in (1). We want to show that $\succeq$ is more split averse than $\succeq'$. To see this, take any wallet x and an amount c. We need to show that $x \succeq [c]$ implies $x \succeq' [c]$.

If $x \succeq [c]$, then $\max\{x_1, \ldots, x_K\} \geq c$. Since v is non-negative and strictly increasing, we directly obtain $\sum_k v(x_k) \geq v(c)$, implying that $x \succeq' [c]$.