NYU 2014 PhD Micro midterm solutions

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Question 1

(0) The DM classifies the lottery outcomes in two categories: good (G) and bad (B). He is optimistic about the good outcomes and pessimistic about the bad outcomes. He evaluates a lottery p by the expected value where he treats all the G (B) outcomes in the support of p as if they are the best (worst) outcomes in the support of p.

(a) Independence: Consider $Z = \{1, 2, 3\}$, $v(x) = x, B = \{1\}$ and $G = \{2, 3\}$. Denote by $u(p)$ the DM's evaluation of the lottery p. The DM is indifferent between the lottery $p = [2]$ and the lottery $q = 0.5[1] + 0.5[3]$. However $u(0.5p + 0.5[3]) = 3 > u(0.5q + 0.5[3]) = u(0.25[1] + 0.75[3]) = 2.5$, which violates independence.

Continuity: [3] is better than [2] but for any $\varepsilon > 0$, $\varepsilon[3]$+(1-ε)2 is indifferent to and not better than [3].

(b) If the support of two lotteries is the same, then the best G-outcome and the worst B-outcome are the same in both lotteries p_1 and p_2. Hence, the comparison between the lotteries is the same as the comparison of the aggregate probabilities they assign to good outcomes. Now also supp$(\alpha p_1 + (1-\alpha)p_3) =$ supp$(\alpha p_2 + (1-\alpha)p_3) = S$. Thus,

$$\begin{aligned}
\alpha p_1 + (1-\alpha)p_3 &\succsim \alpha p_2 + (1-\alpha)p_3 \text{ iff} \\
[\alpha p_1 + (1-\alpha)p_3](S \cap G) &\geq [\alpha p_2 + (1-\alpha)p_3](S \cap G) \text{ iff} \\
p_1(\text{supp}(p_1) \cap G) &\geq p_2(\text{supp}(p_1) \cap G) \text{ iff} \\
p_1 &\succsim p_2
\end{aligned}$$

(c) Monotonicity: If $[a] \succ [b]$ then if p and q are two lotteries such that $p(a) > q(a)$, $p(b) < q(b)$ and $p(x) = q(x)$ for any other x then $p \succsim q$.

The proof is quite trivial, as:

(i) the best outcome in $supp(p) \cap G$ is at least as good as the best outcome in $supp(q) \cap G$.

(ii) The worst outcome in $supp(p) \cap B$ is at least as good as the worst outcome in $supp(q) \cap B$.

(iii) $p(supp(p) \cap G) \geq q(supp(q) \cap G)$.

Question 2

Throughout the question, (a, b, c) is written instead of $(a, b, c) \in B$. Let $X = \{x_1, .., x_n\}$.

(1) A betweenness relation which violates (A1) and satisfies the rest:

$$(x_i, x_j, x_k) \text{ if } i < j \text{ and } i < k$$

(2) A betweenness relation which violates (A2) and satisfies the rest:

Let $n = 4$ and place the four points on a circle $(x_1, x_2, x_3, x_4, x_1)$. For any three points one of them is naturally between the two other.

That is B contains $(x_1, x_2, x_3), (x_2, x_3, x_4), (x_3, x_4, x_1), (x_4, x_1, x_2)$ as well as their symmetric counterparts.

(It is easy to extend it to an example with more than 4 elements by "placing the other elements on a line starting from x_4).

(3) A betweenness relation which violates (A3) and satisfies the rest is the "indifference" relation:

$$B = \{(a, b, c) | a, b, c \text{ are distinct}\}$$

(b) The proof is by induction on n, the cardinaility of X.

For $n = 3$. (A1) and (A3) imply that we can label the elements in X such that $B = \{(a, b, c), (c, b, a)\}$.

Then set $\alpha(a) < \alpha(b) < \alpha(c)$ and we are done.

For the inductive step, suppose the statement holds for some n and let us prove it for $|X| = n + 1$.

Let $y \in X$. By the inductive hypothesis, there is a function α which represnts the relation B on $X - \{y\}$. Enumerate $X - \{y\} = \{x_1, .., x_n\}$ where $\alpha(x_i) < \alpha(x_j)$ iff $i < j$. So for each $1 \leq i < j < k \leq n$, $(x_i, x_j, x_k) \in B$.

Our goal is to extend the finction α by defining $\alpha(y)$ such that it will reprsent B also for triples of elements involving y.

For each $k \in \{1, .., n-1\}$, (A3) and (A1) imply that exactly one of the following holds:

$$(x_k, y, x_{k+1}), (x_k, x_{k+1}, y), (y, x_k, x_{k+1})$$

Suppose (x_k, y, x_{k+1}) for some k. Then by (A2) for all $i < j \leq k$, we have (x_i, x_j, y), for all $k+1 \leq i < j$, we have (y, x_i, x_j) and for all $i \leq k, j \geq k+1$, we have (x_i, y, x_j). Thus, setting $\alpha(y)$ between $\alpha(x_k)$ and $\alpha(x_{k+1})$ works.

Claim: Suppose that (x_k, y, x_{k+1}) holds for no k. Then either (y, x_1, x_2) or (x_{n-1}, x_n, y) is true but not both.

This is suffcient since if (y, x_1, x_2), then for all $1 \leq k < l \leq n-1$, we have (y, x_k, x_l). So setting $\alpha(y) < \alpha(x_1)$ works. Similarly, if (x_{n-1}, x_n, y), then setting $\alpha(y) > \alpha(x_n)$ works.

Proof of Claim: The statements (y, x_1, x_2) and (x_{n-1}, x_n, y) cannot both hold because by (A2), (y, x_1, x_2) implies (y, x_{n-1}, x_n) in contradiction with (x_{n-1}, x_n, y).

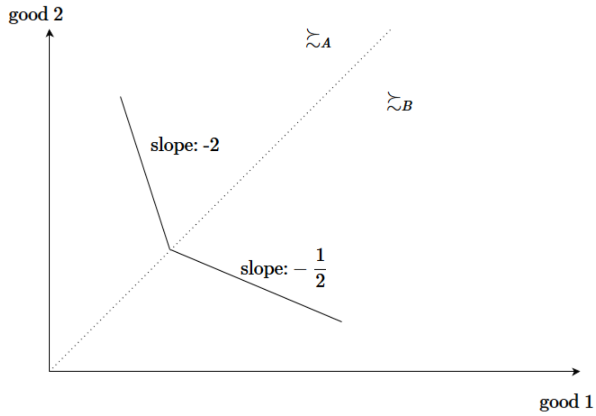
It is left to argue that either (y, x_1, x_2) or (x_{n-1}, x_n, y). Suppose not. Then (x_1, x_2, y) and (y, x_{n-1}, x_n). Thus, there must be a maximal $k < n-1$ for which (x_k, x_{k+1}, y). Then (y, x_{k+1}, x_{k+2}).

To get a contradiction we will check the three possibilities for the relation on the triplet x_k, x_{k+2}, y.

- If (x_k, x_{k+2}, y), then from (x_k, x_{k+1}, x_{k+2}) and (A2), we have (x_{k+1}, x_{k+2}, y). By (A3), this contradicts (y, x_{k+1}, x_{k+2}).
- If (y, x_k, x_{k+2}), then from (x_k, x_{k+1}, x_{k+2}) and (A2), we have (y, x_k, x_{k+1}). By (A3), this contradicts (x_k, x_{k+1}, y).
- If (x_k, y, x_{k+2}), then from (y, x_{k+1}, x_{k+2}) and (A2), we get (x_k, y, x_{k+1}): a contradiction.

Question 3

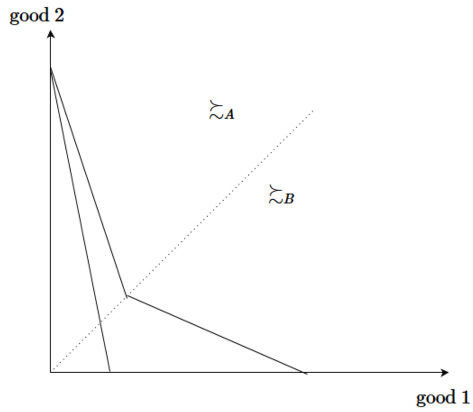
(a) You can rationalise the demand function with the following preferences, the indifference curves of which are drawn below.



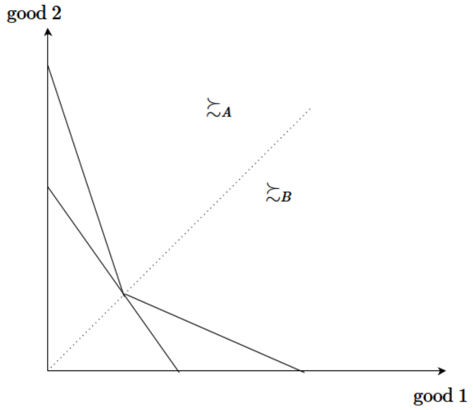
The part of the indifference curve above the diagonal is given by consultant A's indifference curve and the part below the diagonal is given by consultant B's indifference curve. Formally, for any bundle x, let $(d_A(x), d_A(x))$ be the unique point on the diagonal which satisfies $(d_A(x), d_A(x)) \sim_A x$ and similarly for consultant B. Then preferences can be represented by a utility function

$$u(x) = \begin{cases} d_A(x) & \text{if } x_1 \leq x_2 \\ d_B(x) & \text{if } x_1 > x_2 \end{cases}$$

To argue these preferences are consistent with the demand function, consider first $p_1 > 2p_2$. Then both consultants recommend spending the entire budget on good 2. This is also the u-maximal bundle in the budget set as depicted below.

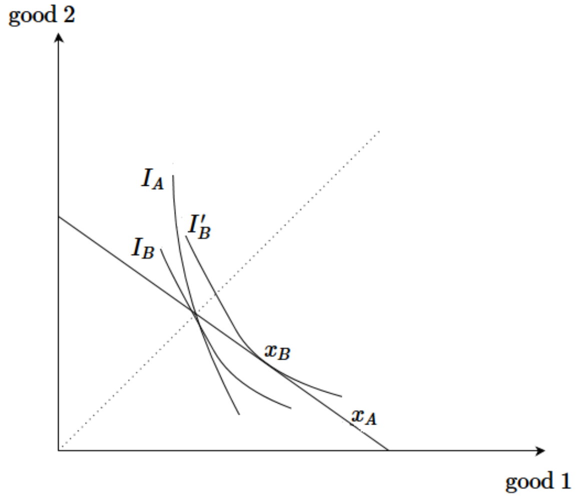


The case $p_2 > 2p_2$ follows analogously. We are left with the case where no good is more than twice expensive than the other. Then consultant A recommends spending everything on good 1 and consultant B recommends spending everything on good 2, so the demand function picks the intersection point of the diagonal and the budget line. This is also the u-maximal bundle in the budget set.



(b) We will use the preferences from part (a) to rationalise the demand function in the general case as well. Let (d, d) denote the intersection point of the diagonal and the budget line.

Firstly, suppose $\frac{p_1}{p_2} \leq \frac{v_B^1(d,d)}{v_B^2(d,d)}$. In this case both consultants' indifference curves I_A and I_B are steeper than the budget line at (d, d). Therefore, the consultants recommend points x_A and x_B below the diagonal. The demand function picks x_B as it is closer to the diagonal. Now the highest attainable indifference curve for the consumer is the one which coincides with I'_B below the diagonal. Hence, the preference-maximising bundle is x_B, as required.



We proceed similarly if $\frac{p_1}{p_2} \geq \frac{v_A^1(d,d)}{v_A^2(d,d)}$.

Finally, suppose $\frac{v_A^1(d,d)}{v_A^2(d,d)} \leq \frac{p_1}{p_2} \leq \frac{v_B^1(d,d)}{v_B^2(d,d)}$. Then A recommends x_A below the diagonal and B recommends x_B above the diagonal. Thus, the demand functions picks (d, d). This is also the preference-maximal bundle because the highest attainable indifference curve consists of I_A above the diagonal and I_B below the diagonal.