Exam: **Microeconomics I, New York University**

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SOLUTION

Question 1

A consumer operates in a world with K commodities. He has in mind a list of consumption priorities, a sequence (k_n, q_n) where $k_n \in \{1,\ldots,K\}$ is a commodity and q_n is a quantity. When facing a budget set (p,w) he purchases the goods according to the order of priorities in the list, until his budget is exhausted. (In the case that his money is exhausted during the n'th stage he purchases whatever proportion of the quantity q_n that he can afford).

(i) How does the demand for the k'th commodity responds to the p_k, p_j $(j \neq k)$ and w?

(ii) Suggest an increasing utility function which rationalizes the consumer's behavior.

(iii) Using the utility function you suggested in (ii) prove the Roy equality for this consumer at (p,w) where the consumer exhausts his entire budget while satisfying his n'th goal.

Solution:

(i) Let y^n be the bundle which gets the value 0 for all commodities besides k_n and the value q_n for commodity k_n.

Given a budget set (p,w) let L be minimum l for which $\sum_{n=1}^{l} p_n q_n > w$. Then

$$x_k(p,w) = \begin{cases} \sum_{n=1}^{L-1} \boldsymbol{y}_k^n & \text{if } k \neq k_L \\ \sum_{n=1}^{L-1} \boldsymbol{y}_k^n + \frac{w - \sum_{n=1}^{L-1} p_{k_n} q_n}{p_k} & \text{if } k = k_L \end{cases}$$

In other words to demand is the highest feasible bundle on the line which connects the bundles

$0, y^1, y^1 + y^2, y^1 + y^2 + y^3 \ldots$

Increasing w weakly increases the demand to each of the commodities. Increasing each of the prices (weakly) decreases the demand to each of the goods.

(ii) Define $u(x)$ to be the largest number t for which $\sum_{n=1}^{[t]} \boldsymbol{y}^n + (t-[t])y^{[t]+1} \leq x$ (where the inequality is inequality of vectors and $[t]$ is the greatest integer of t.) In other words, $u(x)$ is the number of tasks (could be 6.3) that can be fulfilled with the assets in x.

Clearly the consumer's behavior is derived from the maximization of this utility function.

(iii) The indirect utility function $v(p,w)$ which is induced from this utility function is the number of stages which could be obtained given the income w and the price vector p.

Assume that at (p,w) the consumer is in the midst of the n'th stage. Changing the price of good k by a small ε changes the expense of the $k'th$ commodity by $\varepsilon x_k(p,w)$ and thus changes the stage of he can obtain by $-(\varepsilon x_k(p,w)/_{q_n p_n})$

Thus, $\frac{\partial v}{\partial p_k}(p,w) = -x_k(p,w)/_{q_n p_n}$.

A change in ε in the wealth allows a change of $(\varepsilon/_{q_n p_n})$ in his indirect utility and thus

Thus, $\frac{\partial v}{\partial w}(p,w) = 1/_{q_n p_n}$. It follows that:

$$-\frac{\frac{\partial v}{\partial p_k}(p,w)}{\frac{\partial v}{\partial w}(p,w)} = x_k(p,w)$$

Question 2

Consider a decision maker in the world of lotteries, with $Z = R$ being monetary prizes. The decision maker assigns a number $v(z)$ to each amount of money z. The function v is continuous and increasing. The decision maker evaluates each lottery p according to:

$U(p) = \alpha[max\{v(z)|z \in supp(p)\}] + (1-\alpha)[\min\{v(z)|z \in supp(p)\}]$.

(a) Characterize the decision makers of this type who are "risk averse".

(b) Show that if two decision makers of this type, with $\alpha = 1/2$, hold the functions v_1 and v_2 and $v_1 \circ v_2^{-1}$ is concave, then decision maker 1 is more risk averse than decision maker 2.

(c) Do at home: Assume that the two decision makers use $\alpha = 1/2$. Is the concavity of $v_1 \circ v_2^{-1}$ a necessary condition for decision maker 1 to be more risk averse than decision maker 2.

Solution:

(a) Assume $0 < \alpha \leq 1$. Fix a, c such that $a < c$. For any α we can find a number $b \in (a,c)$ such that $\alpha v(c) + (1-\alpha)v(a) > v(b)$. Let α be a number such that $\alpha c + (1-\alpha)a < b$. Consider the lottery p which receives the value c with probability λ and the value a with probability $(1-\lambda)$. Then, $U_\alpha([Ep]) = v(Ep) < v(b) < U_\alpha(p) = \alpha v(c) + (1-\alpha)v(a)$. Thus, $\succsim$ does not exhibit risk aversion.

If $\alpha = 0$, then whatever $v(z)$ is the relation $\succsim$ is risk averse since always $U_0(p) = min_{z \in supp(p)} v(z) \leq v(Ep)$.

(b) Assume that $p \succsim_1 c$. Let $v_1(M) = \max\{v_1(z) \mid z \in supp(p)\}$ and $v_1(m) = \min\{v_1(z) \mid z \in supp(p)\}$. Then, $[v_1(M) + v_1(m)]/2 \geq v_1(c)$. Since $\varphi = v_2 \circ v_1^{-1}$ is convex, then

$$[v_2(M) + v_2(m)]/2 = [\varphi(v_1(M)) + \varphi(v_1(m))]/2) \geq \varphi([v_1(M) + v_1(m)]/2) \geq \varphi(v_1(c)) = v_2(c)$$

That is, $p \succsim_2 c$. ■

Question 3 (Based on Rubinstein (1980))

An individual is comparing pairs of alternatives within a finite set X (|X|≥ 3). His comparison yields unambiguous results, such that either x is evaluated to be better than y (denoted $x \to y$) or y is evaluated to be better than x ($y \to x$). A ranking method assigns to each such relation $\to$ (namely, complete, irreflexive and antisymmetric relation) a preference relation $\succsim(\to)$ over X. Consider the following axioms with respect to ranking methods:

Neutrality: "the names of the alternatives are immaterial". (Formally, let σ be a permutation of X and let $\sigma(\to)$ be the relation defined by $\sigma(x)\sigma(\to)\sigma(y)$ iff $x \to y$. Then, $x \succsim (\to)y$ iff $\sigma(x) \succsim (\sigma(\to))\sigma(y)$.)

Monotonicity: if $x \succsim (\to)y$, then $x \succsim (\to')y$ where $\to'$, differs from $\to$ only in the existence of one alternative z such that $z \to x$ and $x \to' z$.

Independence: The ranking between any two alternatives depends only on the results of comparisons that involve at least one of the two alternatives.

(i) Define $N_\to(x) = |\{z|x \to z\}|$ (the number of alternatives beaten by x). Explain why the scoring method defined by $x \succsim (\to)y$ if $N_\to(x) \geq N_\to(y)$ satisfies the three axioms.

(ii) For each of the properties, give an example of a ranking method which satsifies the other two properties but not that one.

(iii) Prove that the above scoring methods is the only one that satisfies the three properties.

(In the exam you can make do with a proof for a 4-element set X).

Answer:

(i) Neutrality: For any permutation σ of X, $x \succsim y \Leftrightarrow \sigma(x) \succsim (\sigma)\sigma(y)$ since the numbers of victories for x and y in $\to$ are the same as for $\sigma(x)$ and $\sigma(y)$ accordingly in $\to'$.

Monotonicity: If x has at least as many points as y in $\to$, then it will have strictly more victories in $\to'$ where it wins in one additional comparison (which it lost in $\to$).

Independence: The comparison between x and y depends only on the comparisons involving x and y.

(ii) The three axioms are independent.

(a) Consider a method that assigns to any $\to$ the same arbitrary fixed preference relation. This ranking method satisfies Monotonicity and Independence but violates Neutrality.

(b) Consider the ranking method according to $x \succsim (\to) y$ if $N_{\to}(x) \leq N_{\to}(y)$. It satisfies Neutrality and Independence but violates Monotonicity.

(c) Consider the ranking method defined by

$$x \succsim (\to) y \text{ if } \sum_{z \mid x \to z} (N_{\to}(z) + 1) \geq \sum_{z \mid y \to z} (N_{\to}(z) + 1)$$

It satisfies Neutrality and Monotonicity but violates Independence.

(iii)

We will first prove first the following :

Lemma: If a ranking method $\succsim$ satisfies Neutrality and Independence and if $N_{\to}(x) = N_{\to}(y)$, then $x \sim (\to) y$.

Define $A \equiv \{z \mid x \to z \text{ and } z \to y\}$ and $B \equiv \{z \mid z \to x \text{ and } y \to z\}$.

Since $x \to y$ and $N_{\to}(x) = N_{\to}(y)$ we have $|B| = |A| + 1$.

We shall prove by induction on $|A|$: Assume $|A| = 0$.

Let b be the unique element in B. The relation $\to$ cycles on $\{x, y, b\}$. By Independence we can assume that every element which beats both x and y beats b and every element beaten by both x and y is beaten by b. Then, by Neutrality $x \sim y \sim b$.

Assume that the induction hypothesis holds for $|A| = m - 1 \geq 0$ and let $|A| = m$.

Choose $a \in A$ and $b \in B$. By Independence, the ranking of x and y will not change if we assume that (i) $a \to b$ and (ii) $a \to z$ iff $x \to z$ for all $z \notin \{a, b, x, y\}$.

Then $x \to a$, $a \to b$ and $b \to x$, while for all $z \notin \{x, a, b\}$, $x \to z$ iff $a \to z$. It follows from the case in which $|A| = 0$, that $a \sim (\to) x$.

We also have $a \to y$ and $N_{\to}(a) = N_{\to}(y)$. The set $\{z | a \to z \text{ and } z \to y\}$ includes $m - 2$ elements. Therefore, by the induction hypothesis $a \sim (\to) y$. Thus $x \sim (\to) y$.

Finally, consider a ranking method that satisfies the three axioms on the set X. Suppose $x, y \in X$.

If $N_{\to}(x) > N_{\to}(y) \geq 1$, let E be a subset not including y whose $|\{z \mid x \to z\}| - |\{z \mid y \to z\}|$ elements are all beaten by x. Let $\to'$ denote the relation obtained from $\to$ by reversing the results of the comparisons of x to the elements in E. Then $x \sim (\to')y$ by the Lemma. Applying the Monotonicity axiom $|E|$ times, we have $x \succ y$.

If $N_{\to}(x) > N_{\to}(y) = 0$, let h be a third element. Let $\to'$ be the relation derived from $\to$ by reversing the results of the comaprison between y and h. Then, $N_{\to'}(x) \geq N_{\to'}(y) \geq 1$, and therefore $x \succsim (\to')y$ from the previous conclusion. If $y \succsim (\to)x$, then by Monotonicity we would have $y \succ (\to')x$ and hence $x \succ (\to)y$. ■