Course: **Microeconomics, New York University**

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Exam Solutions: **Mid-term, October 2011**

Question 1

Let X be a set of alternatives.

Decision maker of type A uses the following choice procedure. He has a subset of "satisfactory alternatives" in mind. When there are satisfactory elements in the set, he is happy to choose any satisfactory alternative which comes to his mind. If there are none, he is happy with any of the non-satisfactory alternatives.

A decision maker of type B has in mind a set of strict orderings. Whenever he chooses from a set, he is happy with any alternative that is the maxima of at least one ordering.

(a) Define formally the two types of decision makers as **choice correspondences**.

(b) Show that any decision maker of type A can also be described as a decision maker of type B.

(c) Show that there is a decision maker of type B who cannot be described as a decision maker of type A.

Answer

(a)

A: Let S be a set of "satisfactory alternatives"and let $N = X - S$ be the set of "non-satisfactory alternatives".

$$C^S(A) = A \cap S \; if A \cap S \neq \emptyset$$
$$A \quad if otherwise$$

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B: Let $\Lambda = \{\succ_i\}$ be a set of strict orderings.

$$C^\Lambda(A) = \left\{a \in A \;:\; for some \succ_i \;\in \Lambda \;we \;\; have \;\; a \succ_i y \quad \forall y \in A \backslash \{a\}\right\}$$

(b) **CLAIM: Any type A DM can be described as a type B DM**

Let C^S be a choice correspondence of type A characterized by a set S of satisfactory elements.

Let Λ be the set of all orderings which place all elements in S above all elements in $X - S$. Let C^{Λ} be the choice correspondence of type B DM who has Λ in mind. We need to prove that $C^S(A) = C^{\Lambda}(A)$ for all $A \subseteq X$.

1. $C^S(A) \subseteq C^{\Lambda}(A)$.

Let $a \in C^S(A)$. Either $a \in S$ or $a \in N$.

If $a \in S$, then a is the maximum in A of an ordering in Λ which places a first.

If $a \in N$, a is the maximum in A of an ordering in Λ which puts a above all elements in $X - S$.

Hence, $a \in C^{\Lambda}(A)$.

2. $C^S(A) \supseteq C^{\Lambda}(A)$.

Let $a \in C^{\Lambda}(A)$.

If $a \in S$, $a \in C^S(A)$.

If $a \in N$, then there is no element from S in A since if there was, a would not be a maximum of any of the relations in Λ. Thus, $a \in C^S(A)$.

(C)

Consider the following example:

$$X = \{a, b, c\},\ \Lambda = \{\succ_1, \succ_2\}$$

where

$$a \succ_1 b \succ_1 c$$
$$c \succ_2 a \succ_2 b$$

Suppose this DM can be represented as type A. Under this set of strict orderings, $C^{\Lambda}(\{a,b\}) = \{a\}$ which implies that $a \in S \,\&\, b \in N$. Moreover, $C^{\Lambda}(\{a,c\}) = \{a,c\}$ and then $S = \{a,c\}$ and $N = \{b\}$. However, $C^{\Lambda}(\{c,b\}) = \{c,b\}$, where $C^S(\{c,b\}) = \{c\}$.

Question 2

A middleman buys and sells K commodities. He is able to transfer goods between markets 1 and 2 where the price vectors are p^1 and p^2 respectively. A transaction is characterized by $(t_1,..,t_K) \in R^K$ where $t_k > 0$ is a transfer of t_k units of commodity k from market 1 to market 2 and $t_k < 0$ is a transfer of $|t_k|$ units of commodity k from market 2 to market 1. Let T be a set of transactions that he can make. Assume that T is compact and convex.

(a) Formulate the middleman problem if he is a maximizer of profits.

(b) Compare the behavior of the middleman for the two pairs of price vectors (p^1,p^2) and (q^1,q^2) which differ only in $p_k^2 > q_k^2$ (for one commodity k).

(c) Assume that T is the constraint whereby the total number of all goods that the middleman can transfer cannot exceed 100. What can you say about the solution of the middleman problem? (No formal proof is necessary.)

(d) Assume that 0 is an interior point of T. What is the necessary and sufficient condition for no trade to be optimal? (No formal proof is necessary)

Answer

(a)

$$\max_{t \in T} t(p^2 - p^1) \qquad \#$$

Denote the solution by $t(p^1,p^2)$.

(b) Let $p_k^2 - q_k^2 = \varepsilon > 0$. For all (p^1,p^2,q^1,q^2) we have:

$$t(p^1,p^2).(p^2 - p^1) \geq t(q^1,q^2).(p^2 - p^1)$$
$$t(q^1,q^2).(q^2 - q^1) \geq t(p^1,p^2).(q^2 - q^1)$$

The two inequalities imply that

$$[(p^2 - p^1) - (q^2 - q^1)][t(p^1,p^2) - t(q^1,q^2)] \geq 0$$

$$[(p^2 - p^1) - (q^2 - q^1)][t(p^1,p^2) - t(q^1,q^2)] = (0,\ldots,+\varepsilon,\ldots,0)[t(p^1,p^2) - t(q^1,q^2)]$$
$$= +\varepsilon[t_k(p^1,p^2) - t_k(q^1,q^2)] \geq 0$$

and thus $t_k(p^1,p^2) \geq t_k(q^1,q^2)$.

(c) Now

$$T = \left\{ t \in \mathbb{R}^K \ : \ \sum_{k=1}^{K} |t_k| \leq 100 \right\}$$

In this case, the middleman will transfer 100 units of the commodity with the highest price difference from the market with lower price to the market with higher price.

(d)

Given that $0 \in int(T)$, no trade is optimal if and only if $p^1 = p^2$

Question 3

A decision maker forms preferences over "cycles". In each period he is in one of the two states in the set $Z = \{A, B\}$. A *cycle* will be an arbitrary finite sequence of elements in Z. The cycle $(z_1, \ldots, z_L)$ is interpreted as an infinite sequence without a beginning and without an end i.e.. $\ldots.(z_1, \ldots, z_L)(z_1, \ldots, z_L)(z_1, \ldots, z_L)\ldots.$.

With this interpretation in mind, we will assume that **preference relations satisfy the following two properties:**

Invariance to Description: $(z_1, \ldots, z_L) \sim (z_2, \ldots, z_L, z_1)$ for all $(z_1, \ldots, z_L)$.

Duplication: $(z_1, \ldots, z_L) \sim (z_1, \ldots, z_L, z_1, \ldots, z_L, \ldots, z_1, \ldots, z_L)$ for all $(z_1, \ldots, z_L)$.

In this question, we will also discuss the following three properties:

Symmetry: $(z_1, \ldots, z_L) \sim (z_{\sigma(1)}, \ldots, z_{\sigma(L)})$ for all $(z_1, \ldots, z_L)$ and for any permutation σ.

Strong Monotonicity: $(y_1, \ldots, A, \ldots, y_L) \succ (y_1, \ldots, B, \ldots, y_L)$ for all $(y_1, \ldots., y_L)$.

Cancellation: $(y_1, \ldots, y_L) \succsim (z_1, \ldots, z_M)$ and $y_L = z_M$ implies that $(y_1, \ldots, y_{L-1}) \succsim (z_1, \ldots, z_{M-1})$

(a) Construct (no proofs are needed) three examples of preferences over cycles, satisfying Invariance to Description and Duplication. One of them should satisfy Symmetry two should **not** satisfy Symmetry.

(b) What do you think about preferences which satisfy Invariance to Description, Duplication, Strong Monotonicity and Cancellation?

(c) State and prove a proposition regarding the set of preferences which satisfy Invariance to Description, Duplication, Symmetry and Strong Monotonicity.

Answer

(a) The following are three utility functions which represent three preference relations (satisfying the two basic properties). The first also satisfies Symmetry while the last two do not.

1. $u(y_1, \ldots, y_L) = proportion\ of A's\, in\, (y_1, \ldots, y_L)$
2. $u(y_1, \ldots, y_L)$ =the proportion of $y_i = y_{i+1}$ (where $y_{L+1} = y_1$).
3. $u(y_1, \ldots, y_L)$ = the length of the longest sequence $y_i = y_{i+1} = \ldots y_j$ (again $y_{L+1} = y_1$)

(b) Assume there exists a preference relation satisfying the cancellation property and strong monotonicity.

By Duplication $ABAB \sim AB$ which implies by Cancellation that $ABA \sim A$

By Duplication, $A \sim AAA$ and thus $ABA \sim AAA$

However, by strong monotonicity $AAA \succ ABA$. A contradiction.

(c) **Claim**: Suppose $\succsim$ satisfy symmetry and strong monotonicity. Then $y = (y_1, \ldots, y_L) \succsim z = (z_1, \ldots, z_M)$ if and only if $\frac{A_y}{L} \geq \frac{A_z}{M}$, where A_y is the number of states equal to A in $(y_1, \ldots, y_L)$ and A_z is the number of states equal to A in $(z_1, \ldots, z_M)$.

Proof: Let $N(y_1, .., y_L)$ be the sequence of length NL $(y_1, \ldots, y_L, \ldots.., y_1, \ldots, y_L)$. By the duplication property $My \sim y$, and $Lz \sim z$. The two cycles My and Lz are of the same length LM. By symmetry, we can rearrange the duplicated cycles such that the A's are the first elements in the cycle and the B's are the last. The number of A's in My and Lz is MA_y and LA_z, respectively. By strong monotonicity, $My \succsim Lz$ if and only if $MA_y \geq LA_z$. Therefore, $y \succsim z$ if and only if $\frac{A_y}{L} \geq \frac{A_z}{M}$.