Course: Microeconomics, New York University
Lecturer: Ariel Rubinstein
Exam: Mid-term, October 2008
Time: 3 hours (no extensions)
Instructions: Answer the following three questions (each question in a seperate exam booklet)

Question 1

A decision maker has a preference relation over the pairs (x_{me}, x_{him}) with the interpretation that x_{me} is an amount of money he will get and x_{him} is the amount of money another person will get. Assume that

(i) for all (a, b) such that $a > b$ the decision maker strictly prefers (a, b) over (b, a).

(ii) if $a' > a$ then $(a', b) \succ (a, b)$.

The decision maker has to allocate M between him and another person.

(a) *Show that these assumptions guarantee that he will never allocate to the other person more than he gives to himself*

Let $B(M) = \{(a, b)|a + b \leq M\}$ be the set of feasible allocations, and $x(M) = (x_{me}, x_{him})$ be the chosen allocation from the feasible set.
Then $x(M) \succsim (a, b)$ for any $(a, b) \in B(M)$.
Assume $x(M) = (x_1, x_2)$, and $x_1 < x_2$. Then $(x_2, x_1) \succ (x_1, x_2)$ by (i), and (x_2, x_1) is feasible, a contradiction. Therefore $x_1 \geq x_2$.

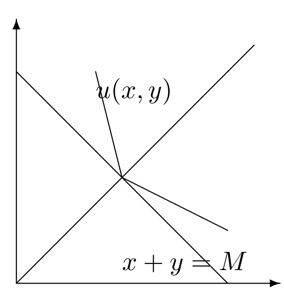
(b) *Assume (i), (ii) and*

(iii) The decision maker is indifferent between (a, a) and $(a - \varepsilon, a + 4\varepsilon)$ for all a and $\varepsilon > 0$.
Show that nevertheless he might allocate the money equally.

Suppose the preferences are represented by, for example

$$u(x, y) = \begin{cases} 4x + y & \text{if } y \geq x \\ 2x + 3y \, * & \text{if } x \geq y \end{cases}$$

* any preferences which have a kink at $x = y$ and satisfy (i) and (iii) will work - see picture



These preferences satisfy monotonicity.
The preferences satisfy (iii) since $\forall a$, $u(a,a) = 5a = 4(a-\varepsilon) + (a+4\varepsilon) = u(a-\varepsilon, a+4\varepsilon)$.
They also satisfy (i) since if $x_2 > x_1$, then $u(x_1, x_2) = 4x_1 + x_2 = 5x_1 + (x_2 - x_1) < 5x + 2(x_2 - x_1) = 2x_2 + 3x_1 = u(x_2, x_1)$.

In this case, the utility would be maximized by setting $x = y = \frac{M}{2}$.

(c) *Assume (i), (ii), (iii) and*

(iv) The decision maker's preferences are also differentiable (according to the definition given in class).

Show that in this case, he will allocate to himself (strictly) more than to the other.

Assume by contradiction the DM chooses to allocate $x(M) = (\frac{M}{2}, \frac{M}{2})$.
By differentiability of the preferences, $v(x) = (4,1)$ (i.e. the hyperplane separating the improving directions would be the hyperplane $4x + y = M$). But then $(1,-1)$ is a strictly improving direction, and the bundle $(\frac{M}{2} + \epsilon, \frac{M}{2} - \epsilon) \succ (\frac{M}{2}, \frac{M}{2}) = x(M)$ would be affordable for any $\epsilon > 0$, a contradiction.

Question 2 (based on work of Kfir Eliaz and Ariel Rubinstein)
Let X be a (finite) set of alternatives. Given any choice problem A (where $|A| \geq 2$), the decision maker chooses a set $D(A) \subseteq A$ of ***two*** *alternatives which he wants to examine more carefully before making the final decision.*
The following are two properties of D:

A1: *If $a \in D(A)$ and $a \in B \subset A$ then $a \in D(B)$.*

A2: *If $D(A) = \{x, y\}$ and $a \in D(A - \{x\})$ for some a different than x and y, then $a \in D(A - \{y\})$.*

Answer the following four questions. A full proof is required only for the last question:

(a) *Find an example of a D function which satisfies both A1 and A2.*

Let $\succ$ be a strict preference on X. Let $D(A)$ be the set of the two $\succ$–best elements in A.

(b) *Find a function D which satisfies A1 and not A2.*

Let $\succ$ be a strict preference on X. Let $D(A)$ be the set containing the $\succ$–best element and the $\succ$–worst element in A.

(c) *Find a function D which satisfies A2 and not A1.*

Let $\succ$ be a strict preference on X. Let $D(A)$ be the set containing the second and third $\succ$–best elements in A (if $|A| \geq 3$) and $D(A) = A$ otherwise.

(d) *Characterize the set of D functions which satisfy both axioms.*

Claim: For any $D(A)$ Satisfying A1, A2 there exists an ordering $\succ$ of the elements of X s.t. $D(A)$ is the set of the two $\succ$–best elements in A.

Proof: We will build $\succ$ inductively. Let $D(X) = \{x_1, x_2\}$.
Define $x_1 \succ x_2 \succ z$ for any $z \in X \backslash \{x_1, x_2\}$.
Assume we defined $x_1 \succ x_2 \succ x_3 \succ \ldots \succ x_k \succ z$ for any $z \in X \backslash \{x_1, \ldots, x_k\}$, and $D(X \backslash \{x_1, \ldots, x_{j-2}\}) = (x_{j-1}, x_j)$ for $3 \leq j \leq k$. By A1, $x_k \in D(X \backslash \{x_1, \ldots, x_{k-1}\})$. Denote x_{k+1} such that $D(X \backslash \{x_1, \ldots, x_{k-1}\}) = \{x_k, x_{k+1}\}$. Define $x_k \succ x_{k+1} \succ z$ for any $z \in X \backslash \{x_1, \ldots, x_{k+1}\}$. This procedure is well defined by A2 (since it guarantees x_k is always well defined). The procedure ends because X is finite, therefore $\succ$ is complete. By construction it is transitive.

Question 3

An economic agent has to choose between projects. The outcome of each project is uncertain. It might yield a failure or one of K "types of success". Thus, each project z can be described by a vector of K non-negative numbers, $(z_1, ..., z_K)$ where z_k stands for the probability that the project success will be of type k.

Let $Z \subset \Re^K_+$ be the set of feasible projects. Assume Z is compact, convex and satsifies "free disposal".

The decision maker is an Expected Utility maximizer.

Denote by u_k the vNM utility from the k–th type of success, and attach 0 to failure. Thus the decision maker chooses a project (vector) $z \in Z$ in order to maximize $\sum z_k u_k$.

(a) *First, formalize the decision maker's problem. Then, formalize (and prove) the claim: If the decision maker suddenly values type k success higher than before, he would choose a project assigning a higher probability to k.*

The DM solves:

$$max_{z \in Z} z \cdot u = max_{z \in Z} \sum_{k=1}^{K} z_k u_k$$

Claim: Let $u'_i = u_i$ for every $i \neq k$, $u'_k > u_k$. Then $z(k) \leq z'(k)$ (where $z(k)$ is the probability assigned to success type k in the project chosen when the valuations are u_k, $z'(k)$ the probabilty assigned when the valuations are u'_k).

Proof: Equivalent to the proof of "The Law of Demand (or Supply)".
Let z be the chosen project under the valuations u_k, z' be the chosen project under u'_k.
Then $(z - z') \cdot [u - u'] = [z - z'] \cdot u + [z' - z] \cdot u' \geq 0$, since $z \cdot u \geq z' \cdot u$, and $z' \cdot u' \geq z \cdot u'$. Since $u - u' = (0, \ldots, 0, u_k - u'_k, 0, \ldots, 0) < 0$, then $z(k) - z'(k) < 0$ QED.

(b) *Apparently, the decision maker realizes that there is an additional uncertainty. The world may go "one way or another". With probability α the vNM utility of the k'th type of success will be u_k and with probability $1-\alpha$ it will v_k. Failure remains 0 in both contingencies.*

First, formalize the decision maker's new problem. Then, formalize (and prove) the claim: Even if the decision maker would obtain the same expected utility, would he have known in advance the direction of the world, the existence of uncertainty makes him (at least weakly) less happy.

The DM now solves:

$$max_{z \in Z} z \cdot [\alpha u + (1-\alpha)v] = max_{z \in Z} \sum_{k=1}^{K} z_k [\alpha u_k + (1-\alpha) v_k]$$

Claim: The maximal expected utility in the uncertain world is weakly less than the maximal expected utility when the direction of the world is known.
Proof: Denote the DM's chosen project under the first direction of the world (with vNM utility u_k) as $z_u = (z_u(1), \ldots, z_u(K)) \in Z$, and his chosen project under the second direction of the world (with vNM utility v_k) $z_v = (z_v(1), \ldots, z_v(2)) \in Z$.
Then $max_{z \in Z} z \cdot [\alpha u + (1-\alpha)v] \leq \alpha [max_{z' \in Z} z' \cdot u] + (1-\alpha)[max_{z' \in Z} z' \cdot v] = \alpha z_u \cdot u + (1-\alpha) z_v \cdot v$. Even if $z_u \cdot u = z_v \cdot v = EU$, then $max_{z \in Z} z \cdot [\alpha u + (1-\alpha)v] \leq EU$.