Solution to the 2007 Exam

Microeconomic Theory I - Ariel Rubinstein

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1. *A consumer in a world of K commodities maximizes the utility function $u(x) = \sum_k x_k^2$.*

   (a) *Calculate the consumer's demand function (whenever it is uniquely defined).*
   Demand is uniquely defined when there is a unique minimal price p_k. The consumer will set $x_k(p, w) = \frac{w}{p_k}$ and $x_j(p, w) = 0$ for all $j \neq k$.

   (b) *Give another preference relation (not just a monotonic transformation of u) which induces the same demand function.*
   Preferences represented by $v(x) = \sum_k x_k$.

   (c) *For the original utility function u, calculate the indirect preferences for $K = 2$. What is the relationship between the indirect preferences and the demand function? (It is sufficient to answer for the domain where $p_1 < p_2$.)*
   Indirect preferences are represented by $v(p, w) = \left(\frac{w}{\min\{p_1,p_2\}}\right)^2$. Roy's Equality represents the relationship. When $p_1 < p_2$, demand can be verified as

   $$x(p,w) = \left(-\frac{\partial v(p,w)/\partial p_1}{\partial v(p,w)/\partial w}, -\frac{\partial v(p,w)/\partial p_2}{\partial v(p,w)/\partial w}\right) = \left(-\frac{-2w/p_1^2}{2/p_1}, 0\right) = \left(\frac{w}{p_1}, 0\right).$$

   (d) *Are the preferences in (a) differentiable (according to the definition given in class)?*
   No. Let $K = 2$ and consider the bundle $(1, 1)$. The vectors $d = (1, -1)$ and $d' = (-1, 1)$ are both improvement directions, but for any set of values $v = v((1, 1))$, if $v \cdot d > 0$, then $v \cdot d' < 0$.

2. *Let X be a set and C be a choice correspondence defined on all non-empty subsets of X. We say that C satisfies Path independence (PI) if for every two disjoint sets A and B, we have $C(A \cup B) = C(C(A) \cup C(B))$. We say that C satisfies Extension (E) if $x \in A$ and $x \in C(\{x, y\})$ for every $y \in A$ implies that $x \in C(A)$ for all sets A.*

(a) *Interpret PI and E.*

PI: The agent's choice set is identical, regardless if he (1) chooses directly from the set A or (2) first partitions A into two subsets, chooses from each of the subsets and then makes a subsequent choice from these two choice sets.

E: If $x \in A$ is chosen when compared (pairwise) with every other alternative in A, then x is in the choice set of A.

(b) *Show that if C satisfies both PI and E, then there exists a binary relation $\succsim$ that is complete, reflexive and satisfies $x \succ y$ and $y \succ z$ implies $x \succ z$, such that $C(A) = \{x \in A \mid \text{for no } y \in A \text{ is } y \succ x\}$.*

For all $x, y \in X$, define $x \succsim y$ if $x \in C(\{x, y\})$. Clearly, $\succsim$ is complete and reflexive. By contradiction, assume $x \succ y$ and $y \succ z$, but $z \succsim x$. Then $z \in C(\{x, z\})$. Define $A = \{x, y, z\}$, and note that PI implies that

$$C(A) = C(C(\{x,y\}) \cup C(\{z\})) = C(\{x,z\}) \Rightarrow z \in C(A) \text{ and}$$

$$C(A) = C(C(\{x\}) \cup C(\{y,z\})) = C(\{x,y\}) \Rightarrow z \notin C(A),$$

a contradiction. Therefore, $x \succ z$.

Finally, if $x \succsim y$ for all $y \in A$, then $x \in C(A)$ by E, and thus $C(A) \supseteq \{x \in A \mid \text{ for no } y \in A, y \succ x\}$. Conversely, if there exists a $y \in A$ such that $y \succ x$, then by PI

$$C(A) = C(C(\{x,y\}) \cup C(A \setminus \{x,y\})) = C(\{y\} \cup C(A \setminus \{x,y\})),$$

and thus $x \notin C(A)$. Thus, $C(A) = \{x \in A \mid \text{for no } y \in A, y \succ x\}$.

(c) *Give one example of a choice correspondence: either one satisfying PI but not E, or one satisfying E but not PI.*

Let $\succsim$ be an ordering over X.

PI, not E: Define $C(A)$ to equal the best and second best elements in A according to $\succsim$.
Satisfies PI: The best and second best elements of $A \cup B$ are contained in $C(A) \cup C(B)$, and thus $C(C(A) \cup C(B))$ equals these first and second best elements in $A \cup B$.
Fails E: Every $x \in A$ is in $C(\{x, y\})$ for all $y \in A$. Nevertheless, only the two best elements in A are included in $C(A)$.

E, not PI: Define $C(A)$ to be the best element in A if $A \neq X$, and $C(X) = X$.
Satisfies E: If $x \in C(\{x, y\})$ for all $y \in A$, then there is no $y \succ x$, and thus $x \in C(A)$.
Fails PI: Let A and B be a partition of X. Then $C(X) \neq C(C(A) \cup C(B))$.

3. *Identify a professor's lifetime with the interval* $[0,1]$. *There are* $K+1$ *academic ranks,* $0,..,K$. *All professors start at rank* 0 *and eventually reach rank* K *sooner or later. Define a career as a sequence* $t = (t_1, ..., t_K)$ *where* $t_0 = 0 \le t_1 \le t_2 \le ... \le t_K \le 1$ *with the interpretation that* t_k *is the time of the* k*'th promotion. (Note that a professor can receive multiple promotions at the same time.) Denote by* $\succsim$ *the professor's preferences on the set of all possible careers.*

   For any $\epsilon > 0$ *and for any career* t *such that* $t_K \le 1 - \epsilon$ *define* $t + \epsilon$ *to be the career* $(t+\epsilon)_k = t_k + \epsilon$ *(all promotions are delayed by* ϵ*). Here are two properties of the professor's preferences:*

   Monotonicity: For any two careers t *and* s*, if* $t_k \le s_k$ *for all* k *then* $t \succsim s$ *and if* $t_k < s_k$ *for all* k *then* $t \succ s$.

   Invariance: For every $\epsilon > 0$ *and every two careers* t *and* s *for which* $t + \epsilon$ *and* $s + \epsilon$ *are well defined* $t \succsim s$ *iff* $t + \epsilon \succsim s + \epsilon$.

   (a) *Formulate the set* L *of careers in which a professor receives all* K *promotions at the same time. Show that if* $\succsim$ *satisfy continuity and monotonicity, then for every career* t *there is a career* $s \in L$ *such that* $s \sim t$.

   $L = \{s \mid s = (\alpha, ..., \alpha) \text{ for some } \alpha \in [0,1]\}$.

   By monotonicity, $(0, ..., 0) \succsim t \succsim (1, ..., 1)$. By continuity, there exists a bundle s on the interval connecting $(0, ..., 0)$ and $(1, ..., 1)$ such that $s \sim t$. Clearly, $s \in L$ since it is on the main diagonal.

   (b) *Show that any preference which is represented by the function* $U(t) = -\sum \Delta_k t_k$ *(for some* $\Delta_k > 0$*) satisfies Monotonicity, Invariance and Continuity.*

   MON: If $t_k \le s_k$ for all k, then $-\Delta_k t_k \ge -\Delta_k s_k$ for all k. Consequently, $U(t) = \sum -\Delta_k t_k \ge \sum -\Delta_k s_k = U(s)$, and thus $t \succsim s$, and analogously in the strict case.

   INV: $t \succsim s \iff -\sum \Delta_k t_k \ge -\sum \Delta_k s_k \iff -\sum \Delta_k t_k - \epsilon \sum \Delta_k \ge -\sum \Delta_k s_k - \epsilon \sum \Delta_k \iff -\sum \Delta_k (t_k + \epsilon) \ge -\sum \Delta_k (s_k + \epsilon) \iff t + \epsilon \succsim s + \epsilon$.

   CON: U is continuous, so $\succsim$ is continuous.

   (c) *One professor evaluates a career by the maximum length of time he had to wait for a promotion, and the smaller this number the better. Show that these preferences cannot be represented by the utility function described in (b).*

   $(.2, .8) \sim (.1, .7)$, and thus preferences fail monotonicity.