ANSWER KEY OF EXAM06

1(a) Maximization the preferences $\succsim$ represented by the Cobb-Douglas utility function $u(x) = x_1^{0.5}x_2^{0.5}$ leads to the given demand function $x(p,w,c)$. There are two different cases to consider.

Case 1: $0.5\frac{w}{p_1} \le c$.

In this case, $x(p,w,c)$ is equal to the point $(0.5\frac{w}{p_1}, 0.5\frac{w}{p_2})$, which is the unique maximizer of u on the usual budget set $B(p,w) = \{x \mid px \le w\}$. In particular, $x(p,w,c)$ is the unique maximizer of u on $B(p,w,c)$.

Case 2: $0.5\frac{w}{p_1} > c$.

Suppose by contradiction that $x(p,w,c) = (c, \frac{w-p_1c}{p_2})$ is not the unique maximizer of u on $B(p,w,c)$. Then there is a $y \in B(p,w,c)$ with $y \neq x(p,w,c)$ such that $y \succsim x(p,w,c)$. By strict monotonicity of u we must have $y_1 < c$, otherwise we would have $x(p,w,c) \succ y$ and moreover we can assume $py = w$. Define $\bar{x} = (0.5\frac{w}{p_1}, 0.5\frac{w}{p_2})$. Now, since $y_1 < c < 0.5\frac{w}{p_1}$, $x(p,w,c)$ can be written as a strict convex combination of the points y and $\bar{x}$. Since $\bar{x}$ is the unique maximizer of u on $B(p,w)$, by strict convexity of $\succsim$ we must have $x(p,w,c) \succ y$, a contradiction.

1(b) Suppose by contradiction that $x(p,w,c)$ is consistent with maximization of a preference relation $\succsim$. Fix a price vector p and wealth level w. Pick a $c < \frac{w}{p_1}$, so that $x(p,w,c) = (0.5c, \frac{w-p_1 0.5c}{p_2})$. Now we can pick a c', sufficiently close to c, such that

$$0.5c' < c < c' < \frac{w}{p_1}.$$

Since $0.5c' < \frac{w}{p_1}$, this implies $x(p,w,c') = (0.5c', \frac{w-p_1 0.5c'}{p_2}) \neq x(p,w,c)$. Moreover, since $0.5c < c'$, we have $x(p,w,c) \in B(p,w,c')$, and hence, $x(p,w,c') \succ x(p,w,c)$. On the other hand, since $0.5c' < c$, $x(p,w,c') \in B(p,w,c)$, and hence, $x(p,w,c') \prec x(p,w,c)$, a contradiction.

1(c) Fix a parameter vector $t^* = (p^*, w^*, c^*)$ and assume $\frac{\partial V(t^*)}{\partial c} > 0$. We claim that $x_1(t^*) = c^*$.

Suppose by contradiction that $x_1(t^*) < c^*$. But then since $\frac{\partial V(t^*)}{\partial c} > 0$, there exists an $\varepsilon > 0$ such that $V(p^*, w^*, c^* - \varepsilon) < V(t^*)$ and $c^* - \varepsilon > x_1(t^*)$. Since $p^*x(t^*) = w^*$, it follows that $x(t^*) \in B(p^*, w^*, c^* - \varepsilon)$. Hence, $V(t^*) = u(x(t^*)) \le V(p^*, w^*, c^* - \varepsilon)$, a contradiction.

2(a)

- Let $c^*(A,a) \equiv a$.
- Let $\succsim$ be a strict ordering on X and let $c^*(A,a)$ be the $\succsim$ maximal element of A.

2(b) Let us write "DB" and "EIIA" instead of "Default bias" and "Extended IIA", respectively.

Asymmetry: If $x \succ y$, then $c^*(\{x,y\},y) = x$. So, by DB, $c^*(\{x,y\},x) = x \neq y$, that is, "not $y \succ x$".

Transitivity: Suppose $x \succ y$ and $y \succ z$. Then, $c^*(\{x,y\},y) = x$ and $c^*(\{z,y\},z) = y$.

Now, $c^*(\{x,y,z\},z)$ cannot be y, otherwise by DB, we would have $c^*(\{x,y,z\},y) = y$, and from EIIA it would follow that $c^*(\{x,y\},y) = y$.

$c^*(\{x,y,z\},z)$ cannot be z either, otherwise by EIIA, we would have $c^*(\{z,y\},z) = z$.

So, $c^*(\{x,y,z\},z)$ must be x. By EIIA, this implies $c^*(\{x,z\},z) = x$, that is, $x \succ z$.

2(c) As in the first example of part (a), we may have $c^*(\{x,y\},y) = y$ and $c^*(\{x,y\},x) = x$.

2(d) If $a \in A$ is not $\succ$ maximal in A, for some $y \in A$ with $y \neq a$ we have $y \succ a$, that is, $c^*(\{a,y\},a) = y$. Suppose by contradiction a belongs to $C(A)$, that is, $c^*(A,x) = a$ for some $x \in A$. Then, by DB, $c^*(A,a) = a$. So, by EIIA, $c^*(\{a,y\},a) = a$, a contradiction which shows that $a \notin C(A)$.

Conversely, assume $a \in A$ is $\succ$ maximal in A. We claim that $c^*(A,a) = a$. Suppose by contradiction that $c^*(A,a) = y \neq a$. Then EIIA implies $c^*(\{a,y\},a) = y$, that is, $y \succ a$. This contradicts maximality of a and proves that $c^*(A,a) = a$. By definition of $C(A)$, we immediately conclude that $a \in C(A)$.

The interpretation here is that the choice correspondence C chooses those elements which are not $\succ$ dominated by some other feasible alternative.

3(a) Suppose $U(x) = \sum_k \frac{x_k}{n(x)} v_k$.

A1: For any natural number λ

$$U(\lambda x) = \sum_k \frac{\lambda x_k}{n(\lambda x)} v_k = \sum_k \frac{\lambda x_k}{\lambda n(x)} v_k = U(x).$$

A2: If $n(x) = n(y)$, then

$$U(x) \geq U(y) \Leftrightarrow \sum_k x_k v_k \geq \sum_k y_k v_k \Leftrightarrow$$

$$\sum_k (x_k + z_k) v_k \geq \sum_k (y_k + z_k) v_k \Leftrightarrow U(x+z) \geq U(y+z),$$

where in the last equivalence we use the fact that $n(x+z) = n(y+z)$.

3(b) Suppose that the decision maker chooses a ball from bag x, and he gets a prize depending on the color of the ball he chose. Then we can identify a bag x with a lottery $p(x)$ in which the prize associated with the color x is received with probability $x/n(x)$. Since $p(x) = p(\lambda x)$ we should expect $x \sim \lambda x$. Moreover, A2 would follow in this case from the independence axiom.

3(c) Let $K = 3$, and let $\succsim_L$ be the usual lexicographic relation on $\mathbb{R}^3$. Define the preference relation $\succsim$ on X as

$$x \succsim y \Leftrightarrow \left(\frac{x_1}{n(x)}, \frac{x_2}{n(x)}, \frac{x_3}{n(x)}\right) \succsim_L \left(\frac{y_1}{n(y)}, \frac{y_2}{n(y)}, \frac{y_3}{n(y)}\right).$$

A1: Note that for any x and any natural number λ, $\frac{x_i}{n(x)} = \frac{\lambda x_i}{n(\lambda x_i)}$ for all i. Hence, $x \sim \lambda x$.

A2: $x \succsim y$ and $n(x) = n(y)$ iff $(x_1, x_2, x_3) \succsim_L (y_1, y_2, y_3)$ iff $(x_1 + z_1, x_2 + z_2, x_3 + z_3) \succsim_L (y_1 + z_1, y_2 + z_2, y_3 + z_3)$ iff $x + z \succsim y + z$ (since $n(x+z) = n(y+z)$).

To see that $\succsim$ does not admit a representation of the form given in part (a), suppose to the contrary that $U(x) = \sum_{i=1}^{3} \frac{x_k}{n(x)} v_k$ represents $\succsim$.

Note that $(1,0,n) \succ (0,1,0)$ for all n. Hence, $\frac{v_1}{n+1} + \frac{n v_3}{n+1} > v_2$ for all n. Passing to limit as $n \to \infty$ gives $v_3 \geq v_2$. On the other hand, $(0,1,0) \succ (0,0,1)$ implies $v_2 > v_3$, a contradiction.