Microeconomics 501 - Midterm Exam Solutions

1. **Problem 1 (Time Preferences)**

   Let $X = \mathcal{R}^+ \times \{0, 1, 2, ...\}$ (where (x, t) is interpreted as getting \$$x$ at time t). Assume that a decision maker has a preference relation on this space with the following properties:

   - *He is indifferent between getting \$0 at time 0, or at any other time*
   - *For any positive amount of money he prefers to get it as soon as possible*
   - *He likes money*
   - *His preference between (x, t) and $(y, t+1)$ is independent of t (interpret it)*
   - *Continuity*

   (a) *Define the continuity assumption for this model.*

   Since time is discrete, "continuity" refers to "continuity in x". That is: for any (x, t) and $(\widehat{x}, \widehat{t})$ with $(x, t) \succ (\widehat{x}, \widehat{t})$, there are balls $B(x)$ and $\widehat{B}(\widehat{x})$ such that for any $x' \in B(x)$ and $x'' \in \widehat{B}(\widehat{x})$, $(x', t) \succ (x'', \widehat{t})$.

   Equivalently: for any sequences $(x^n, t) \to (x, t)$ and $(\widehat{x}^n, \widehat{t}) \to (\widehat{x}, t)$ with $(x^n, t) \succsim (\widehat{x}^n, \widehat{t})$ for all n, $(x, t) \succsim (\widehat{x}, \widehat{t})$.

   Equivalently: for any pair (x, t), the sets $\{x' \in \mathcal{R}^+ | (x', t) \succsim (x, t)\}$ and $\{x' \in \mathcal{R}^+ | (x, t) \succsim (x', t)\}$ are closed.

   (b) *Show that any preference relation satisfying the above assumptions has a utility representation.*

   Claim 1: For any pair (x, t), there is a number $v(x, t) \in \mathcal{R}^+$ such that $(x, t) \sim (v(x, t), 0)$.

   Proof. Pick any pair (x, t) with $x \neq 0$, and define the sets

$$\begin{aligned} \overline{X}(x,t) &\equiv \{x' \in \mathcal{R}^+ | (x', 0) \succsim (x, t)\} \\ \underline{X}(x,t) &\equiv \{x' \in \mathcal{R}^+ | (x, t) \succsim (x', 0)\} \end{aligned}$$

   First, observe that the sets are non-empty. (Properties 1 and 3, together with the assumption $x > 0$, imply that $0 \in \underline{X}(x, t)$; i.e., $(x, t) \succsim (0, t) \sim (0, 0)$. Property 2 implies that $x \in \overline{X}(x, t)$; i.e., $(x, 0) \succsim (x, t)$). By continuity, $\overline{X}(x, t)$ and $\underline{X}(x, t)$ are closed. Then since $\mathcal{R}^+ = \overline{X}(x, t) \cup \underline{X}(x, t)$ is connected, it must be that $\overline{X}(x, t), \underline{X}(x, t)$ have a non-empty intersection. Any element $x^* \in \overline{X}(x, t) \cap \underline{X}(x, t)$ (by definition) satisfies $(x, t) \sim (x^*, 0)$. Assuming that property 3 means that he "strictly" likes money, x^* is unique:[1] define $v(x, t) \equiv x^*$.

   If $x = 0$, then set $v(0, t) = 0$. (Property 1 says that $(0, t) \sim (0, 0)$ for all t; property 3 then implies that for any $x > 0$, $(0, t) \sim (0, 0) \precsim (x, 0)$; hence, $v(0, t) \equiv 0$ is the only value satisfying $(v(0, t), 0) \sim (0, t)$). ■

   Claim 2: The preference relation is represented by $u(x, t) = v(x, t)$.

[1] If monotonocity is weak, so that there are several elements x^* with $(x, t) \sim (x^*, 0)$, just pick one.

Proof.

$$\begin{aligned} u(x,t) &\geq u(x',t') \Leftrightarrow v(x,t) \geq v(x',t') \\ &\Leftrightarrow (v(x,t),0) \succsim (v(x',t'),0) \text{ (by Property 3)} \\ &\Leftrightarrow (x,t) \succsim (x',t') \text{ (by definition of } v(), \text{ using transitivity)} \end{aligned}$$

■

(c) *Verify that a preference relation which has the form $u(x)\delta^t$ (with $\delta < 1, u(0) = 0, u$ continuous and increasing) satisfies all axioms.*

Assume that $v(x,t) \equiv u(x)\delta^t$ represents the preferences.

Axiom 1: For all t, t', $v(0,t) = u(0)\delta^t = 0 = u(0)\delta^{t'} = v(0,t')$. So, since $v()$ represents $\succsim$, $(0,t) \sim (0,t')$.

Axiom 2: For any $t < t'$, $v(x,t) = \delta^t u(x) > \delta^{t'} u(x) = v(x,t')$; so, $(x,t) \succ (x,t')$.

Axiom 3: This holds since $u(\cdot)$ is increasing

Axiom 4: For all t, t',

$$\begin{aligned} (x,t) \succsim (y,t+1) &\Leftrightarrow \delta^t[u(x) - \delta u(y)] > 0 \\ &\Leftrightarrow \delta^{t'}[u(x) - \delta u(y)] > 0 \\ &\Leftrightarrow (x,t') \succsim (y,t'+1) \end{aligned}$$

Axiom 5: Continuity in x follows from the fact that $u(\cdot)$ is continuous.

(d) *Formulate a concept "one preference is more impatient than another preference".*

One way to say this: for two preferences $\succsim^1$ and $\succsim^2$, $\succsim^1$ is more impatient than $\succsim^2$ if for any (x,t), and any (x',t') with $t' < t, x' < x$,

$$(x',t') \succsim^2 (x,t) \Rightarrow (x',t') \succsim^1 (x,t)$$

(i.e. - whenever "type 2" is willing to give up some consumption in exchange for getting it earlier, so is the more impatient "type 1").

(e) *Discuss a claim that a preference represented by $u_1(x)\delta_1^t$ is more impatient than a preference represented by $u_2(x)\delta_2^t$ iff $\delta_1 < \delta_2$.*

This would be clear with $u_1(\cdot) = u_2(\cdot) = u(\cdot)$: then for $(x',t') < (x,t)$,

$$\begin{aligned} \delta_2^{t'} u(x') \geq \delta_2^t u(x) &\Leftrightarrow x' \geq u^{-1}\left(\delta_2^{(t-t')} u(x)\right) \\ &\geq u^{-1}\left(\delta_1^{(t-t')} u(x)\right) \\ &\Leftrightarrow u(x') \geq \delta_1^{(t-t')} u(x) \\ &\Leftrightarrow \delta_1^{t'} u(x') \geq \delta_1^t u(x) \\ &\Leftrightarrow (x',t') \succsim^1 (x,t) \end{aligned}$$

However, the claim is not necessarily true if $u_1 \neq u_2$. For instance, suppose $u_1(x) = \ln x$, $u_2(x) = x$, $\delta_1 = \frac{1}{2}$, and $\delta_2 = \frac{1}{\sqrt{3}} > \frac{1}{2}$. Then person 2 is indifferent between getting \$1 at time 0 and \$3 at time 2, since

$$\delta_2^2 u_2(3) = 1 = \delta_2^0 u_2(1)$$

However, person 1, the supposedly more impatient (lower δ) guy, prefers to wait for the \$3:

$$\delta_1^2 u_1(3) - \delta_1^0 u_1(1) = \frac{1}{4}\ln 3 - \ln 1 = \frac{1}{4}\ln 3 > 0$$

2. **Indirect Utility Functions**

Discuss the following consumer. The consumer's initial wealth is w. He likes as much money as possible but for survival he must buy one and only one unit of one and only one of the goods denoted 1,...,K. Commodity k's price is p_k and all prices are less than w (for simplicity, concentrate on a domain where all prices are distinct).

For some reason the consumer prefers not to be seen purchasing the cheapest good, and he always purchases the second cheapest good.

(a) *Define an "indirect utility function" for the consumer.*

If he always purchases the 2nd cheapest good (and this is optimal for him), but only gets "utility" from wealth, then his "indirect utility" is

$$v(w,p) = w - p_{i^*}$$

where i^* is the second cheapest good. (More generally, any increasing function of $(w - p_{i^*})$ would do, and would not change results below).

(b) *Study the properties of the indirect utility function: monotonicity, continuity, and convexity in prices.*

<u>Monotonicity:</u> The indirect utility function satisfies weak monotonicity in prices (a small increase in p_i^* leads to a strict decrease in $v(w,p)$, provided that all prices are distinct (so that i^* will remain the "2nd cheapest good"); however, small changes in the prices of the remaining goods will not have any effect on indirect utility (again, this relies on all prices being distinct). The function is also monotonic in w.

<u>Continuity:</u> $v(p,w)$ is continuous - if all prices are distinct, then small changes will not change the "2nd cheapest good" i^*; then continuity is obvious (since $w - p_{i^*}$ is continuous in both w and p_{i^*}).

<u>Convexity:</u> For convexity, we would need to show that for any p, p', w and any $\alpha \in (0,1)$,

$$\begin{aligned} v(\alpha p + (1-\alpha)p', w) &\leq \alpha v(p,w) + (1-\alpha)v(p',w) \\ \Leftrightarrow \left(\alpha p + (1-\alpha)p'\right)_{i^*} &\geq \alpha p_{j^*} + (1-\alpha)p'_{k^*} \end{aligned}$$

where $(\alpha p + (1-\alpha)p')_{i^*}, p_{j^*}, p'_{k^*}$ are the 2nd cheapest prices in $(\alpha p + (1-\alpha)p'), p, p'$ (respectively). This does not hold. For instance, suppose there are 3 goods, and consider the price vectors $p = (1,3,2), p' = (2.5, 0.2, 2)$; let $\alpha = \frac{1}{2}, w = 4$. Then $p_{j^*} = p_{k^*} = 2$, so $\alpha v(p,w) + (1-\alpha)v(p',w) = 4 - 2 = 2$. However, $(\alpha p + (1-\alpha)p') = (1.75, 1.6, 2)$; the 2nd lowest price here is 1.75, corresponding to an indirect utility of $v(\alpha p + (1-\alpha)p', w) = 2.25$.

(c) *State the "Roy's equality" for this model and explain why it holds (in every price vector where all prices are distinct).*

Roy's equality: $-\dfrac{\partial V/\partial p_i}{\partial V/\partial w} = x_i(p,w)$ (or, the slope of an indifference curve of $V(p,w)$ in $(p_i, w)-$space is $\left.\frac{dw}{dp_i}\right|_{(p,w)} = x_i(p,w)$). To see that this holds, let i^* denote the 2nd cheapest good; again observe that if all prices are distinct, then i^* will remain the 2nd cheapest good for small changes in prices. Hence, differentiating the above indirect utility function (or any increasing function of it) yields

$$\frac{-\partial V/\partial p_i}{\partial V/\partial w} = \begin{cases} 1 \text{ if } i = i^* \\ 0 \text{ otherwise} \end{cases} = x_i(p, w)$$

Explanation without derivatives: a change in p_i only affects your utility if i is the second cheapest commodity. In this case: for a small price change dp_i, you must continue to purchase one unit of good i (for survival), with an expenditure increase of $1 \cdot dp_i$. To compensate you for this change (since you only derive utility from wealth), there must be a wealth increase of $dw = dp_i$. Hence, the slope of the indifference curve of V is

$$\frac{dw}{dp_i} = \begin{cases} 1 \text{ if } i = i^* \\ 0 \text{ if } i \neq i^* \end{cases}$$

3. **Problem 3 (Random dictatorship)**

Consider the aggregation of preference relations defined on the set $\{A, B, L\}$, where L is a lottery which assigns A or B with equal probabilities. Assume that all preference relations satisfy the vNM assumptions.

(a) *Show that there is a social welfare function satisfying the IIA and Pareto axioms which is not dictatorial.*

Consider the majority rule - i.e., for a profile $\{\succsim_i\}$, $x \succ (\{\succsim_i\})\, y$ iff $n_{\succsim}(x; y) > n_{\succsim}(y; x)$, where $n_{\succsim}(x; y)$ is the number of individuals i with $x \succsim_i y$.

Clearly, this satisfies the Pareto axiom - if every individual prefers x to y, then so does society, according to the majority rule. It also satisfies IIA: consider two profiles $\{\succsim_i\}$ and $\{\succsim_i'\}$, and pick two pairs of alternatives (a, b) and (x, y) s.t. $a \succsim_i b$ iff $x \succsim_i' y$. Then

$$\begin{aligned} a \succ \{\succsim_i\}\, b &\Leftrightarrow n_{\succsim}(a; b) > n_{\succsim}(b; a) \\ &\Leftrightarrow n_{\succsim'}(x; y) > n_{\succsim'}(y; x) \\ &\Leftrightarrow x \succ' (\{\succsim_i'\})\, y \end{aligned}$$

It also induces a "well-behaved" social welfare function - i.e., there are no Condorcet cycles or violations of transitivity. To see this, note that if individual $i's$ preferences satisfy the vNM axioms, then there is an expected utility representation $U(\cdot)$. Then $U\left(\frac{1}{2}A + \frac{1}{2}B\right) = \frac{1}{2}U([A]) + \frac{1}{2}U([B])$, so

$$A \succ B \Leftrightarrow A \succ \left(\frac{1}{2}A + \frac{1}{2}B\right) \succ B$$

Thus individual preferences, and hence social preferences, are completely determined by the preferences between any two alternatives. For instance,

$$\begin{aligned} A \succ (\{\succsim_i\}) \left(\frac{1}{2}A + \frac{1}{2}B\right) &\Leftrightarrow n_{\succsim}\left(A; \frac{1}{2}A + \frac{1}{2}B\right) > n_{\succsim}\left(\frac{1}{2}A + \frac{1}{2}B; A\right) \\ &\Leftrightarrow n_{\succsim}\left(\frac{1}{2}A + \frac{1}{2}B; B\right) > n_{\succsim}\left(B; \frac{1}{2}A + \frac{1}{2}B\right) \\ &\Leftrightarrow \left(\frac{1}{2}A + \frac{1}{2}B\right) \succ (\{\succsim_i\})\, B \end{aligned}$$

(where the 2nd line follows from the fact that individual i prefers A to $\frac{1}{2}A+\frac{1}{2}B$ iff he prefers $\frac{1}{2}A+\frac{1}{2}B$ to B; so, $n_{\succsim}(A;\frac{1}{2}A+\frac{1}{2}B)=n_{\succsim}(\frac{1}{2}A+\frac{1}{2}B;B)$).

(b) *Reconcile this fact with Arrow's impossibility theorem.*

Arrow's impossibility theorem says that if there are at least three alternatives, then the only SWF satisfying (in general) the Pareto and IIA axioms is the dictatorship. We already saw in PS10 that this is not necessarily true if the domain of preferences is restricted - for instance, if all preferences are single-peaked. In this case, the domain is essentially restricted to two alternatives, since strict preference for A over B (or vice versa) means that A dominates all other alternatives.

4. **Problem 4 (Choice with a status quo)**

Let $X=R^K$ be a "grand set". Let c be a function which assigns an element in S to every pair (S,d), where S is a closed and convex subset of X and $d\in X$ (d is not necessarily in S). The function c is interpreted as the choice from S given that d is the "status quo".

(a) *Formulate the property of a choice function such that the choice from a set $S\cap T$ given a status quo d can be done invariantly either in one stage or in two stages, by first selecting an element in S given the original status quo, and then selecting a point in T (given the new "status quo").*

The "path independence" property: for any two subsets S,T of X with $S\cap T\neq\emptyset$ and any point $d\in X$,

$$c(S\cap T;d)=c(T;c(S;d))$$

(b) *Show that for $K=2$ there is no such function, whereas such a function exists for $K=1$.*

Claim 1: If $c(\cdot)$ satisfies the property, then $c(S;d)=d$ whenever $d\in S$.

Proof.

$$\begin{aligned} d=c(\{d\};d) &= c(\{d\}\cap S;d) \\ &= c(S,c(\{d\};d)) \quad \text{(by path independence)} \\ &= c(S;d) \end{aligned}$$

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Claim 2: If $K=2$, then there is no choice function satisfying path independence.

Proof. Pick any three elements a,b,d which do not lie on the same straight line. Let $e^*\equiv c([a,b];d)$ (where $[a,b]$ denotes the line segment between a and b). Assume $e^*\neq a$ (which is without loss of generality - otherwise, just change the line segment). We have:

$$\begin{aligned} a &= c(\{a\},d) \\ &= c([a,b]\cap[a,d];d) \\ &= c([a,b];c([a,d];d)) \quad \text{(by path independence)} \\ &= c([a,b];d) \quad \text{(by Claim 1, since } d\in[a,d]) \\ &= e^*, \ \text{a contradiction.} \end{aligned}$$

■

Claim 3: If $K = 1$, then the choice function which always chooses the point in S closest to d, satisfies path independence. (To see this: consider two overlapping line segments $S = [s_1, s_2]$ and $T = [t_1, t_2]$, and take any point d on the real line. Let $c^* = c(S \cap T; d), c_1^* = c(S; d)$, and $c_2^* = c(T; c_1^*)$. If $d \in S \cap T$, then it is clear that $c_1^* = c_2^* = c^* = d$. If not, then assume that $d < \max\{s_1, t_1\}$ (the argument for $d > \min\{s_2, t_2\}$ is symmetric). Then it must be that $c^* = \max\{s_1, t_1\}$;any point further to the left lies outside of $S \cap T$, while any point further to the right is not the closest to $\max\{s_1, t_1\}$. For path independence, we need to show that $c_2^* = \max\{s_1, t_1\}$ (given that d is left of this). If $s_1 < t_1$, then $c_1^* = \left\{ \begin{array}{c} s_1 \text{ if } d < s_1 \\ d \text{ if } d \in [s_1, t_1] \end{array} \right\}$; since this is left of t_1, the closest point to c_1^* in T is $c_2^* = t_1$, as required. If $t_1 < s_1$, $d < s_1$ implies that $c_1^* = s_1$; since $s_1 \in [t_1, t_2]$ by assumption, the closest point on T to s_1 is $c_2^* = s_1$, as required.