**Problem Set 16**

1. Histories are sequences $h = ((x_1^1, x_2^1), (x_1^1, x_2^1), \ldots, (x_1^t, x_2^t))$ of pairs of proposals in $[0,1]$, where $x_i^s$ represents player $i$'s proposal at period $s$ of how much player 1 should receive and $x_1^s \neq x_2^s$ for $s < t$. The finite terminal histories are those $h = ((x_1^1, x_2^1), (x_1^1, x_2^1), \ldots, (x_1^t, x_2^t))$ with $x_1^t = x_2^t$, $u_1(h) = \delta^{t-1} x_1^t$ and $u_2(h) = \delta^{t-1}(1 - x_1^t)$. The infinite histories are terminal histories $h$ where there is no agreement and $u_i(h) = 0$ for $i = 1, 2$.

Any outcome $(x^*, t)$ representing agreement about outcome $x^*$ at period $t$ is a SPE outcome of this game ($t = \infty$ corresponds to disagreement). The following strategies constitute a SPE leading to $(x^*, t)$:

*Player 1:* In all periods $s < t$, propose $x_1^s = 1$. In all periods $s \geq t$, propose $x_1^s = x^*$.

*Player 2:* In all periods $s < t$, propose $x_2^s = 0$. In all periods $s \geq t$, propose $x_2^s = x^*$.

2. Let $x^* = (1, 0)$, $y^* = (1 - c_1, c_1)$, $0 < c_1 < 1$, $c_1 < c_2$ and $u_i(D) = -\infty$ for $i = 1, 2$. Consider the strategies:

*Player 1:* Always offer $x^*$, accept an offer $y$ if and only if $y_1 \geq y_1^* = 1 - c_1$.

*Player 2:* Always offer $y^*$, accept all offers.

*Step 1: The above strategies form a SPE:*

*Optimality for player 1:* In a subgame after which player 1 offers, player 1 gets her maximum possible payoff of 1 by offering $x^*$, so her strategy is optimal. Now consider a subgame after which player 2 offers $y \in X$. Maximum possible that player 1 can get by rejecting is $1 - c_1$, so if $y_1 \geq 1 - c_1$ then it is optimal for her to accept. On the other hand if $y_1 < 1 - c_1$, rejecting now and offering $x^*$ next period yields her a payoff of $1 - c_1$, which is the maximum that she can get by using any other strategy that rejects now and strictly greater than what she can get by accepting now.

*Optimality for player 2:* Consider a subgame after which player 1 offers $x \in X$. Accepting gives player 2 a payoff of $x_2 \geq 2$, whereas if player 2 employs a strategy that rejects $x$ now, then *given player 1's strategy*: $c_1 - c_2$ is the maximum possible that player 2 can get if the strategy leads to agreement in the next period; $0 - 2c_2$ is what player 2 gets if the strategy leads to agreement two periods from now; $c_1 - 3c_2$ is the maximum possible that player 2 can get if the strategy leads to agreement three periods from now; $0 - 4c_2$ is what player 2 gets if the strategy leads to agreement four periods from now; ...; $-\infty$ is what player 2 gets if the strategy never leads to agreement. So by rejecting now, the maximum possible player 2 can get is

$$max\,\{c_1 - c_2, 0 - 2c_2, c_1 - 3c_2, 0 - 4c_2, \ldots, -\infty\} = c_1 - c_2 < 0.$$

Therefore accepting $x$ now is an optimal reply for player 2.

Now consider a subgame after which player 2 offers. Offering $y^*$ now gives player 2 a payoff of $y_2^*$ whereas if player 2 uses a strategy that offers $y$ with $y_2 > y_2^*$, this leads to rejection by player 1 and the most that player 2 can get *given player 1's strategy* is

$$max\,\{0 - c_2, c_1 - 2c_2, 0 - 3c_2, c_1 - 4c_2, \ldots, -\infty\} = -c_2 < 0$$

by arguments similar to above. So offering $y^*$ is optimal for player 2.

*Step 2: The SPE payoffs are unique:*

Let $i$, $j$ be an arbitrary permutation of 1,2. Let $G_i$ denote the subgame where player $i$ is the first to offer, let $m_i$ and $M_i$ denote $i$'s infimum and supremum SPE payoffs in $G_i$, respectively. Then:

$$(a) \quad m_i \geq 1 - max\{M_j - c_j, 0\}$$

and

$$(b) \quad M_i \leq max\,\{1 - max\{m_j - c_j, 0\}, 1 - m_j - c_i\} \leq 1 - (m_j - c_j)$$

Assume that $M_2 \geq c_2$, then $m_1 \geq 1 - (M_2 - c_2)$ by $(a.i = 1)$ and $M_2 \leq 1 - (m_1 - c_1)$ by $(b.i = 2)$, a contradiction to $c_1 < c_2$. So $M_2 < c_2$, i.e. $M_1 = m_1 = 1$ by $(a.i = 1)$. By $(a.i = 2)$, $m_2 \geq 1 - (1 - c_1) = c_1$ and by $(b.i = 2)$, $M_2 \leq 1 - (1 - c_1) = c_1$ so $m_2 = M_2 = c_1$.

*Step 3: The SPE is unique:*

Player 2 accepts any offer because her continuation payoff if she rejects is $c_1 - c_2 < 0$. So it can only be optimal for player 1 to offer $x^*$. Similarly if player 1 rejects, then her continuation payoff is $1 - c_1$, so she would accept any $y$ with $y_1 > 1 - c_1$ and reject any $y$ with $y_1 < 1 - c_1$. It is not optimal for player 2 to offer $x$ that will be rejected by player 1, because in that case her continuation payoff is no more than $-c_2 < 0$. It is also not optimal for player 2 to offer $y$ with $y_1 > 1 - c_1$, so player 2 offers $y^*$ and player 1 accepts.

3. Assume additionally that each $u_i$ is continuous. Then this is Proposition 122.1 in Osborne and Rubinstein.

4. Assume that $\delta_i, \delta \in (0,1)$ and $u_i \geq 0$. Note that $\delta^{t-1} = (\delta_i^{t-1})^{\frac{ln\delta}{ln\delta_i}}$ and define $v_i(x) = (u_i(x))^{\frac{ln\delta}{ln\delta_i}}$. Then $v_i(x)\delta^{t-1} = \left(u_i(x)\delta_i^{t-1}\right)^{\frac{ln\delta}{ln\delta_i}}$ is a monotonic transformation of $u_i(x)\delta_i^{t-1}$ and therefore represents the same time preference over $X \times T$.

5. Let each $u_i$ be continuous, $u_1$ strictly increasing, $u_2$ strictly decreasing on $X = [0,1]$, $u_1(0) = u_2(1) = 0$ and let $u_2 \circ u_1^{-1}\colon [0, u_1(1)] \to [0, u_2(0)]$ be concave.[1] Let $z^*$ maximize $u_1(z)u_2(z)$ over all $z \in X$ and for each $\delta \in (0,1)$, let $x_\delta^*$, $y_\delta^*$ solve:

$$\delta u_1(x_\delta^*) = u_1(y_\delta^*) \qquad \delta u_2(y_\delta^*) = u_2(x_\delta^*).$$

Then $y_\delta^* < x_\delta^*$ and $u_1(x_\delta^*)u_2(x_\delta^*) = u_1(y_\delta^*)u_2(y_\delta^*)$.

We will next show that $y_\delta^* \leq z^* \leq x_\delta^*$. Suppose not, wlog let $z^* < y_\delta^* < x_\delta^*$. Then $u_1(z^*) < u_1(y_\delta^*) < u_1(x_\delta^*)$ and $u_2(z^*) > u_2(y_\delta^*) > u_2(x_\delta^*)$, i.e.:

$$u_1(z^*) = (1+\gamma_\delta)u_1(y_\delta^*) - \gamma_\delta u_1(x_\delta^*), \qquad \text{where} \quad \gamma_\delta = \frac{u_1(y_\delta^*) - u_1(z^*)}{u_1(x_\delta^*) - u_1(y_\delta^*)} \in (0,1).$$

So:

$$u_2(z^*) \leq (1+\gamma_\delta)u_2(y_\delta^*) - \gamma_\delta u_2(x_\delta^*).$$

by concavity of $u_2 \circ u_1^{-1}$. Then:

$$u_1(z^*)u_2(z^*) \leq (1+\gamma_\delta)^2 u_1(y_\delta^*)u_2(y_\delta^*) + \gamma_\delta^2 u_1(x_\delta^*)u_2(x_\delta^*) - \gamma_\delta(1+\gamma_\delta)[u_1(y_\delta^*)u_2(x_\delta^*) + u_2(y_\delta^*)u_1(x_\delta^*)]$$

$$< (1+\gamma_\delta)^2 u_1(y_\delta^*)u_2(y_\delta^*) + \gamma_\delta^2 u_1(x_\delta^*)u_2(x_\delta^*) - \gamma_\delta(1+\gamma_\delta)[u_1(x_\delta^*)u_2(x_\delta^*) + u_1(y_\delta^*)u_2(y_\delta^*)] = u_1(x_\delta^*)u_2(x_\delta^*)$$

where the second inequality follows from $u_1(y_\delta^*)u_2(x_\delta^*) + u_2(y_\delta^*)u_1(x_\delta^*) = u_1(x_\delta^*)u_2(x_\delta^*) + u_1(y_\delta^*)u_2(y_\delta^*) + [u_1(x_\delta^*) - u_1(y_\delta^*)][u_2(y_\delta^*) - u_2(x_\delta^*)] > u_1(x_\delta^*)u_2(x_\delta^*) + u_1(y_\delta^*)u_2(y_\delta^*)$ and the last equality follows from $u_1(x_\delta^*)u_2(x_\delta^*) = u_1(y_\delta^*)u_2(y_\delta^*)$, a contradiction.

So $\delta u_1(x_\delta^*) = u_1(y_\delta^*) \leq u_1(z^*) \leq u_1(x_\delta^*)$, i.e. $u_1(x_\delta^*) \to u_1(z^*)$ as $\delta \to 1$. Since $u_1$ is strictly increasing and continuous we conclude that $x_\delta^* \to z^*$ as $\delta \to 1$.

6. In the following, I use $i$, $j$ for an arbitrary permutation of 1,2. Remember that in the unique SPE of model 5, $i$ offers $\frac{1-\delta_j}{1-\delta_1\delta_2}$ to herself and $\frac{\delta_j(1-\delta_i)}{1-\delta_1\delta_2}$ to $j$ and $i$ accepts an offer $x$ if and only if $x_i \geq \frac{\delta_i(1-\delta_j)}{1-\delta_1\delta_2}$.

[1] For the latter, concavity of $u_1$ and $u_2$ is sufficient but not necessary (e.g. let $u_1(x) = \sqrt{x}$ and $u_2(x) = e^{\frac{1}{2}(1-x)} - 1$, then $u_2$ is not concave but $u_2 \circ u_1^{-1}$ is).

Consider a change in the model where now each player can opt out when responding to an offer. So we only add finite terminal histories $h = (x^1, N, x^2, N, \ldots, x^t, Out)$ with $u_i(h) = d_i^* \delta_i^{t-1}$ to the original model. Everything else (including the player function) is the same. Moreover assume that each player prefers her payoff in the unique SPE of the original model to opting out, i.e. $d_i^* < \frac{\delta_i(1-\delta_j)}{1-\delta_1\delta_2}$ for $i = 1, 2$.

It is straightforward to verify that the above SPE continues to be an SPE in the new model with opting out. Let us show that it is the only one. Let $G_i$ denote the subgame of the new model where player $i$ is the first to offer, let $m_i$ and $M_i$ denote $i$'s infimum and supremum SPE payoffs in $G_i$, respectively. Then:

$$(a) \quad m_i \geq 1 - max\{\delta_j M_j, d_j^*\} \geq 1 - \delta_j M_j$$

where the second inequality follows from $d_j^* < \delta_j \frac{1-\delta_i}{1-\delta_1\delta_2} \leq \delta_j M_j$. Similarly:

$$M_i \leq max\left\{1 - max\{d_j^*, \delta_j m_j\}, \delta_i(1 - m_j)\right\}$$

where $m_j \geq 1 - \delta_i M_i$ by $(a)$, so $\delta_i(1 - m_j) \leq \delta_i^2 M_i$. Since $M_i > 0$ and $\delta_i \in (0, 1)$ the above inequality is equivalent to:

$$(b) \quad M_i \leq 1 - max\{d_j^*, \delta_j m_j\} \leq 1 - \delta_j m_j.$$

Then $(a)$ and $(b)$ imply that $m_i = M_i = \frac{1-\delta_j}{1-\delta_1\delta_2}$ for $i = 1, 2$. Standard arguments show that the unique SPE strategies in the new game are the same as in the original one.

7. Assume that $0 < \beta < 1$. Then these preferences are not time consistent. To see this let $\beta\delta < \alpha < \delta$: at period 1, the agent prefers receiving 1 at period 3 to receiving $\alpha$ at period 2 but at period 2, she prefers receiving $\alpha$ at period 2 to receiving 1 at period 3. Therefore the preferences of the agent over terminal histories is not well defined. One way to analyze such time preferences is to perceive player $i$ as a different agent at each period. Then the new set of agents is $\{(i, t) : i = 1, 2 \;\; t = 1, 2, \ldots\}$, $U_{(i,t)}(x, t) = v_i(x)$, $U_{(i,t)}(x, s) = v_i(x)\beta\delta^{s-t}$ if $s > t$ and $U_{(i,t)}(x, s) = 0$ otherwise. It is easy to check that the old SPE for $\delta_i = \beta\delta$ continues to be an SPE in the modified game.