Problem Set 15 (Ex: 1 and 7)-Extensive Games with Perfect Information

1. *Given an extensive game define a reduced strategy of player i in that game to be a function f_i whose domain is a subset of $\{h \in H\colon P(h) = i\}$ and has the following properties:*

(i) *it associates with every h in the domain of f_i an action in $A(h)$ and*

(ii) *a history h with $P(h) = i$ is in the domain of f_i if and only if all the actions of player i in h are those dictated by f_i (that is, if $h = (a^k)$ and $h' = (a^k)_{k=1,\ldots,L}$ is a subsequence of h with $P(h') = i$ then $f_i(h') = a^{L+1}$).*

Specify exactly the sense in which the set of Nash equilibria of an extensive game with perfect information corresponds to the Nash equilibria of the strategic game in which the set of actions of each player is the set of his reduced strategies.

Let S_i denote the strategies and F_i the reduced strategies of i. For any $s_i \in S_i$, define the s_i-*consistent* histories in $P^{-1}(i)$ by:

$$H_i(s_i) = \left\{ h = (a^k) \in P^{-1}(i) | \text{For any strict subhistory } h' = (a^1, \ldots, a^L) \in P^{-1}(i) : s_i(h') = a^{L+1} \right\}.$$

Then the map S_i to F_i that takes s_i to $s_i|_{H_i(s_i)}$ is well defined (i.e. into F_i) and onto F_i. Note also that the iterative definition of the outcome function works for reduced strategy profiles and for any profile of strategies $(s_i)_{i\in N} \in \prod_{i\in N} S_i$, we have $O\left((s_i)_{i\in N}\right) = O\left(\left(s_i|_{H_i(s_i)}\right)_{i\in N}\right)$. So for any $f^* = (f_i^*)_{i\in N} \in \prod_{i\in N} F_i^*$ and $s^* = (s_i^*)_{i\in N} \in \prod_{i\in N} S_i^*$ such that $f_i^* = s_i^*|_{H_i(s_i)}$ for any $i \in N$, f^* is a NE in reduced strategies if and only if s^* is a NE, i.e.:

$$\forall i \in N, f_i \in F_i : u_i\left(O(f_i^*, f_{-i}^*)\right) \geq u_i\left(O(f_i, f_{-i}^*)\right) \Leftrightarrow \forall i \in N, s_i \in S_i : u_i\left(O(s_i^*, s_{-i}^*)\right) \geq u_i\left(O(s_i, s_{-i}^*)\right).$$

7. *Under the official rules of Chess, a game actually does not end when a position is repeated three times, unless the player who has to move declares a "draw". Thus, Chess is actually not a finite game. Prove that Chess with this additional detail still has a "value".*

Let A denote the set of positions in the above Chess game Γ, let d stand for a draw declaration. Then a history is a (possibly infinite) sequence in $A \cup \{d\}$ that is in accordance with the usual rules of the game and the above additional detail. The game has four possible outcomes: W (white wins), B (black wins), D (there is a draw), N (none of the previous: corresponding to the infinite histories where no one wins nor declares a draw). We assume wlog that $u_i(i) = 1$, $u_i(j) = -1$, $u_i(D) = u_i(N) = 0$ for distinct $i, j \in \{W, B\}$. The game is strictly competitive since $u_W = -u_B$.

We will next construct a NE strategy profile $(s_W, s_B) \in S_W \times S_B$. Let H' denote the set of histories of Γ where at least one position has occurred three or more times and let H'' be those histories in H' that are not preceded by a proper subhistory in H'.[1] For any $i \in \{W, B\}$,

[1] Then H'' consists of those nonterminal histories of finite length $h = (a_0, a_1, \ldots, a_K)$ where $a_0, a_1, \ldots, a_K \in A$, the position a_K occurs exactly three times in h and all other positions occur twice or less. Also note that if $h \in H'$ then any super history h of h' is also in H'.

and a nonterminal history $h \in H'$ such that $i = P(h)$, set $s_i(h) = a$ if i can make a move to a position $a \in A$ where she wins the game at (h, a) and otherwise set $s_i(h) = d$. Note that for $h \in H' \setminus Z$, the restriction of these strategies to $\Gamma(h)$ constitute a NE of the subgame $\Gamma(h)$.

Now consider the truncated game $\tilde{\Gamma}$, where the set of histories is $\tilde{H} = (H \setminus H') \cup H''$. Then the terminal histories of the truncated game are $\tilde{Z} = (Z \setminus H') \cup H''$.[2] We associate the same outcome to terminal nodes in $Z \setminus H'$ as in Γ, and we associate the above specified continuation NE outcome to histories in H'', i.e. for $h \in H''$, if $P(h)$ can make a move to a position $a \in A$ where she wins the game at (h, a), then we associate h with $P(h)$ and otherwise we associate h with D. Everything else (utilities, player function,...) is as in the original game Γ.

The truncated game $\tilde{\Gamma}$ is finite since in any history $h \in \tilde{H}$, no position occurs more than three times. Therefore it has a subgame perfect equilibrium profile $(\tilde{s}_W, \tilde{s}_B)$. Now for any $i \in \{W, B\}$, and a nonterminal node $h \in H \setminus H'$ of the original game Γ with $i = P(h)$, let $s_i(h) = \tilde{s}_i(h)$. This completes the construction of the strategy profile (s_W, s_B) for the original game Γ. To see that (s_W, s_B) is a NE of Γ, let $i, j \in \{W, B\}$ be distinct and $s_i' \in S_i$, then:

$$u_i\left(O\left(s_i, s_j\right)\right) \geq u_i\left(O\left(\left(s_i'|_{P^{-1}(i) \cap \tilde{H} \setminus \tilde{Z}}, s_i|_{P^{-1}(i) \cap H' \setminus Z}\right), s_j\right)\right) \geq u_i\left(O\left(s_i', s_j\right)\right).$$

The first inequality follows from $\left(s_W|_{P^{-1}(W) \cap \tilde{H} \setminus \tilde{Z}}, s_B|_{P^{-1}(B) \cap \tilde{H} \setminus \tilde{Z}}\right) = (\tilde{s}_W, \tilde{s}_B)$ being a NE of the truncated game $\tilde{\Gamma}$, given the continuation strategies $\left(s_W|_{P^{-1}(W) \cap H' \setminus Z}, s_B|_{P^{-1}(B) \cap H' \setminus Z}\right)$. The second inequality follows from $(s_W|_h, s_B|_h)$ being a NE of the subgame $\Gamma(h)$ for any $h \in H' \setminus Z$. So (s_W, s_B) is a NE of Γ. Moreover since Chess is strictly competitive, the equilibrium payoff is unique and any NE strategy of a player guarantees the player her equilibrium payoff.

Reference to a direct approach:

For those interested, Julie pointed out a paper by Schwalbe and Walker (GEB 2001) that summarizes earlier game theoretic work done on chess and proofs of the existence of value that do not use Kuhn's theorem as we did above:

> "...Zermelo then addresses two problems: First, what does it mean for a player to be in a winning position and is it possible to define this in an objective mathematical manner... ...a necessary and sufficient condition is the nonemptiness of a certain set, containing all possible sequences of moves such that a player (say White) wins independently of how the other player (Black) plays. But should this set be empty, the best a player could achieve would be a draw. So he defines another set containing all possible sequences of moves such that a player can postpone his loss for an infinite number of moves, which implies a draw. This set may also be empty, i.e., the player can avoid his loss for only finitely many moves if his opponent plays correctly. But this is equivalent to the opponent being able to force a win..."

[2]Please note the following: (a) $H \setminus H'$ and H'' are disjoint, (b) if $h \in \tilde{H}$ then any subhistory h' of h is also in $\tilde{H}$, (c) $\tilde{Z}$ is indeed the set of terminal histories in $\tilde{H}$ and (d) $\tilde{H} \setminus \tilde{Z} = H \setminus (H' \cup Z)$. Remarks (b) and (c) ensure that the extensive game $\tilde{\Gamma}$ is well-defined.