Exercise 5-Problem Set 14

Consider the following symmetric game:

	L	R
L	$0,0$	$1,2$
R	$2,1$	$0,0$

(a) Find the set of mixed strategy NE and the ESS of this game.

(b) Suppose now that when two players interact, they play the above situation twice and a player can decide about what to do at the second period as a function of what has happened in the first period. Assume that each player maximizes the expected value of the sum of her payoffs in the two stages of the game. Find the set of symmetric mixed strategy NE and ESS.

Part (a):

Let $A = \{L, R\}$ and $\Delta(A)$ be the set of mixed strategies over A. (For $\alpha \in \Delta(A)$ and $a \in A$, $\alpha(a)$ denotes the probability of taking action a and we always associate a particular action with the degenerate probability distribution that leads to it.) Then, $u(\alpha, \beta) = \alpha(L)\beta(R) + 2\alpha(R)\beta(L)$ is the payoff to the player who plays $\alpha \in \Delta(A)$ when the opponent plays $\beta \in \Delta(A)$.

Clearly, the only pure strategy NE of the above game are (L, R) and (R, L). Let $(b^{*1}, b^{*2}) \in \Delta(A) \times \Delta(A)$ be a non pure strategy NE. Wlog let $0 < b^{*1}(L) < 1$. Since player 1 takes both actions with positive probability, b^{*2} should be such that player 1 is indifferent between playing L or R, i.e. $1 - b^{*2}(L) = u(L, b^{*2}) = u(R, b^{*2}) = 2b^{*2}(L)$, so $b^{*2}(L) = \frac{1}{3}$. By a symmetric argument $b^{*1}(L) = \frac{1}{3}$. So the game has three NE: (L, R), (R, L) and (b^*, b^*) where $b^* = \left(\frac{1}{3}L, \frac{2}{3}R\right)$ with associated payoff vectors $(1, 2)$, $(2, 1)$ and $\left(\frac{2}{3}, \frac{2}{3}\right)$ respectively.

Out of these three equilibria (b^*, b^*) is the only symmetric one. To see that it is actually an ESS, let $b \in \Delta(A)$. By construction of b^* $u(L, b^*) = u(R, b^*) = \frac{2}{3}$, so $u(b, b^*) = u(b^*, b^*) = \frac{2}{3}$. Moreover, the term $u(b^*, b) - u(b, b) = [\frac{1}{3}(1 - b(L)) + 2\frac{2}{3}b(L)] - [3b(L)(1 - b(L))] = \frac{1}{3} - 2b(L) + 3b(L)^2$ is uniquely minimized at $b(L) = \frac{1}{3} = b^*(L)$, where it is 0. Therefore $u(b^*, b) > u(b, b)$ when $b \neq b^*$, so b^* is an ESS.

Part (b):

In this repeated game, the set of strategies of a player is $S = \Delta(A) \times \{f | f\colon A \times A \to \Delta(A)\}$. For $\sigma = (\sigma_1, \sigma_2) \in S$, σ_1 denotes the mixed strategy that I play in the first game and for every $(a, b) \in A \times A$, $\sigma_2(a, b)$ denotes the mixed strategy that I play in the second game conditional on the outcome of the first game being "I played a and my opponent played b".

Then, $U(\sigma, \nu) = u(\sigma_1, \nu_1) + \sum_{(a,b)\in A\times A} \sigma_1(a)\nu_1(b)u\left(\sigma_2(a, b), \nu_2(b, a)\right)$ denotes the expected sum of payoffs to the player who plays $\sigma \in S$ when the opponent plays $\nu \in S$.

Part (b.1): The Set of Symmetric Nash Equilibria

Let $\sigma \in S$ be a symmetric NE. Since by best replying to the opponent's strategy a player can guarantee herself at least an expected payoff of $\frac{2}{3}$ in each stage of the game, we should have $U(\sigma, \sigma) \geq \frac{4}{3}$. Moreover, if the probability of an outcome (a, b) has positive probability in the first stage (i.e. $\sigma_1(a)\sigma_1(b) > 0$), then the following on-equilibrium path play in the second stage $(\sigma_2(a, b), \sigma_2(b, a))$ should be a NE of the one shot game in part (a). Otherwise if e.g. $u\left(\alpha, \sigma_2(b, a)\right) > u\left(\sigma_2(a, b), \sigma_2(b, a)\right)$ for some $\alpha \in \Delta(A)$ in the one shot game, then the strategy $\sigma' \in S$ obtained from σ by only changing $\sigma_2(a, b)$ to α is such that $U(\sigma', \sigma) > U(\sigma, \sigma)$, contradicting that σ is a NE.

Note that if $\sigma_1(a) = 1$ for some action $a \in A$, then $u(\sigma_1, \sigma_1) = 0$. Then, as argued in the previous paragraph $\sigma_2(a, a)$ is a symmetric NE of the one shot game, so by part (a) $\sigma_2(a, a) = \left(\frac{1}{3}L, \frac{2}{3}R\right)$. But then $U(\sigma, \sigma) = \frac{2}{3} < \frac{4}{3}$, a contradiction. Therefore, $\sigma_1 \in \Delta(A)$ is a completely mixed strategy, in particular each outcome (a, b) has positive probability in the first stage and therefore $(\sigma_2(a, b), \sigma_2(b, a))$ is one of the three NE of the one shot game in part (a), for any $(a, b) \in A \times A$. Then $\sigma_2(a, a) = \left(\frac{1}{3}L, \frac{2}{3}R\right)$ for $a = L, R$, and there are three possibilities:

1. $\sigma_2(L, R) = L$ and $\sigma_2(R, L) = R$,
2. $\sigma_2(L, R) = R$ and $\sigma_2(R, L) = L$,
3. $\sigma_2(L, R) = \sigma_2(R, L) = \left(\frac{1}{3}L, \frac{2}{3}R\right)$.

We will show that there is a unique symmetric equilibrium of the repeated game associated with each one of these three cases.

Part (b.1.1): We need to find a completely mixed strategy equilibrium in the first period given that both players play according to σ_2 (as specified in case 1), in the second stage game and they both take this into account. Adding up the continuation payoffs this is equivalent to finding a symmetric completely mixed NE σ_1 of the following game:

	L	R
L	$0+\frac{2}{3}, 0+\frac{2}{3}$	$1+1, 2+2$
R	$2+2, 1+1$	$0+\frac{2}{3}, 0+\frac{2}{3}$

$\equiv$

	L	R
L	$\frac{2}{3}, \frac{2}{3}$	$2, 4$
R	$4, 2$	$\frac{2}{3}, \frac{2}{3}$

So $\sigma_1 = \left(\frac{2}{7}L, \frac{5}{7}R\right)$, $\sigma_2(L, R) = L$, $\sigma_2(R, L) = R$ and $\sigma_2(a, a) = \left(\frac{1}{3}L, \frac{2}{3}R\right)$ for $a = L, R$ is a symmetric NE.

Part (b.1.2): In this case, we are looking for a totally mixed symmetric NE σ_1 of the following game:

	L	R
L	$0+\frac{2}{3}, 0+\frac{2}{3}$	$1+2, 2+1$
R	$2+1, 1+2$	$0+\frac{2}{3}, 0+\frac{2}{3}$

$\equiv$

	L	R
L	$\frac{2}{3}, \frac{2}{3}$	$3, 3$
R	$3, 3$	$\frac{2}{3}, \frac{2}{3}$

So $\sigma_1 = \left(\frac{1}{2}L, \frac{1}{2}R\right)$, $\sigma_2(L, R) = R$, $\sigma_2(R, L) = L$ and $\sigma_2(a, a) = \left(\frac{1}{3}L, \frac{2}{3}R\right)$ for $a = L, R$ is a symmetric NE.

Part (b.1.3): Finally, in this case we are looking for a totally mixed symmetric NE σ_1 of the following game:

	L	R
L	$0+\frac{2}{3}, 0+\frac{2}{3}$	$1+\frac{2}{3}, 2+\frac{2}{3}$
R	$2+\frac{2}{3}, 1+\frac{2}{3}$	$0+\frac{2}{3}, 0+\frac{2}{3}$

$\equiv$

	L	R
L	$\frac{2}{3}, \frac{2}{3}$	$\frac{5}{3}, \frac{8}{3}$
R	$\frac{8}{3}, \frac{5}{3}$	$\frac{2}{3}, \frac{2}{3}$

So $\sigma_1 = \left(\frac{1}{3}L, \frac{2}{3}R\right)$, $\sigma_2(L, R) = \sigma_2(R, L) = \left(\frac{1}{3}L, \frac{2}{3}R\right)$ and $\sigma_2(a, a) = \left(\frac{1}{3}L, \frac{2}{3}R\right)$ for $a = L, R$ is a symmetric NE.

Part (b.2): The Set of ESS

We will show that out of the three symmetric equilibria above, the first two are ESS's. Let us first show that last one is not an ESS, let σ^* denote the third strategy above and let σ

denote the second one. Then it is straightforward to compute that $U(\sigma,\sigma^*) = \frac{4}{3} = U(\sigma^*,\sigma^*)$ and $U(\sigma,\sigma) = \frac{11}{6} > \frac{3}{2} = U(\sigma^*,\sigma)$ showing that the third strategy is not an ESS.

Part (b.2.1): The strategy $\sigma^* \in S$ defined by $\sigma_1^* = \left(\frac{2}{7}L, \frac{5}{7}R\right)$, $\sigma_2^*(L,R) = L$, $\sigma_2^*(R,L) = R$ and $\sigma_2^*(a,a) = \left(\frac{1}{3}L, \frac{2}{3}R\right)$ for $a = L, R$ is an ESS.

Let $\sigma \in S$, if $U(\sigma^*,\sigma^*) > U(\sigma,\sigma^*)$ then we are done, so assume that $U(\sigma^*,\sigma^*) = \frac{34}{21} = U(\sigma,\sigma^*)$. This implies that σ is also a best reply to σ^*, i.e. in particular if $\sigma_1(L) > 0$ then $\sigma_2(L,R) = L$ and if $\sigma_1(R) > 0$ then $\sigma_2(R,L) = R$. So, we can compute:

$$U(\sigma^*,\sigma) = \frac{2}{7}(1-\alpha) + 2\frac{5}{7}\alpha + \frac{2}{7}\alpha\left[\frac{1}{3}(1-\beta_L) + 2\frac{2}{3}\beta_L\right] + \frac{2}{7}(1-\alpha) + 2\frac{5}{7}\alpha + \frac{5}{7}(1-\alpha)\left[\frac{1}{3}(1-\beta_R) + 2\frac{2}{3}\beta_R\right],$$

$$U(\sigma,\sigma) = 3\alpha(1-\alpha) + \alpha^2 3\beta_L(1-\beta_L) + 3\alpha(1-\alpha) + (1-\alpha)^2 3\beta_R(1-\beta_R).$$

where $\alpha = \sigma_1(L)$, $\beta_L = \sigma_2(L,L)(L)$ and $\beta_R = \sigma_2(R,R)(L)$. By using the above formulae, it is tedious but straightforward to verify that $U(\sigma^*,\sigma) - U(\sigma,\sigma)$ is uniquely minimized over all $(\alpha,\beta_L,\beta_R) \in [0,1]^3$ at $(\alpha,\beta_L,\beta_R) = \left(\frac{2}{7}, \frac{1}{3}, \frac{1}{3}\right)$ (i.e. at $\sigma = \sigma^*$) where it attains 0.[1] Therefore, $U(\sigma^*,\sigma) > U(\sigma,\sigma)$ if $\sigma \neq \sigma^*$, showing that σ^* is an ESS.

Part (b.2.2): The strategy $\sigma^* \in S$ defined by $\sigma_1^* = \left(\frac{1}{2}L, \frac{1}{2}R\right)$, $\sigma_2^*(L,R) = R$, $\sigma_2^*(R,L) = L$ and $\sigma_2^*(a,a) = \left(\frac{1}{3}L, \frac{2}{3}R\right)$ for $a = L, R$ is an ESS.

Let $\sigma \in S$, if $U(\sigma^*,\sigma^*) > U(\sigma,\sigma^*)$ then we are done, so assume that $U(\sigma^*,\sigma^*) = \frac{11}{6} = U(\sigma,\sigma^*)$. This implies that σ is also a best reply to σ^*, i.e. in particular if $\sigma_1(L) > 0$ then $\sigma_2(L,R) = R$ and if $\sigma_1(R) > 0$ then $\sigma_2(R,L) = L$. So, we can compute:

$$U(\sigma^*,\sigma) = \frac{1}{2}(1-\alpha) + 2\frac{1}{2}\alpha + \frac{1}{2}\alpha\left[\frac{1}{3}(1-\beta_L) + 2\frac{2}{3}\beta_L\right] + 2\frac{1}{2}(1-\alpha) + \frac{1}{2}\alpha + \frac{1}{2}(1-\alpha)\left[\frac{1}{3}(1-\beta_R) + 2\frac{2}{3}\beta_R\right],$$

$$U(\sigma,\sigma) = 3\alpha(1-\alpha) + \alpha^2 3\beta_L(1-\beta_L) + 3\alpha(1-\alpha) + (1-\alpha)^2 3\beta_R(1-\beta_R).$$

where $\alpha = \sigma_1(L)$, $\beta_L = \sigma_2(L,L)(L)$ and $\beta_R = \sigma_2(R,R)(L)$.

By using the above formulae, it can again be verified that $U(\sigma^*,\sigma) - U(\sigma,\sigma)$ is uniquely minimized over all $(\alpha,\beta_L,\beta_R) \in [0,1]^3$ at $(\alpha,\beta_L,\beta_R) = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}\right)$ (i.e. at $\sigma = \sigma^*$) where it attains 0. Therefore, $U(\sigma^*,\sigma) > U(\sigma,\sigma)$ if $\sigma \neq \sigma^*$, showing that σ^* is an ESS.

[1] One way to do this is the following: note that the difference is convex in β_L and β_R for every given α. So you can first minimize the term for every given α, plug the optimal values for β_L and β_R in terms of α and eventually minimize a polynomial in α. A similar argument also works for part b.2.2.