Lecture 16: Modeling Bargaining
Readings: Osborne and Rubinstein Ch 7

Like many game theoretical situations, a bargaining situation is one where there is a mixture of common interests and conflict of interests. The bargainers have common interest to reach an agreement but there is more than one possible mutually beneficial agreements and players differ in their interests regarding the agreement to reach. What is special about a bargaining situation is that in order for the agreement to be implemented there is a need for the common consent of all bargainers.

Bargaining situations can be viewed as the cornerstone to economic models of markets which operate so that exchange is done through pairwise matches in which the terms of exchange are determined through a bargaining game.

More than talking about bargaining itself, this lecture is supposed to demonstrate the use of the extensive game model and especially of the tool of SPE. We will review several models of bargaining, each corresponding to a different "procedure" and assumptions about preferences, mainly in order to demonstrate the modeling considerations.

Note that applying the extensive game model implies that we make strong assumptions about the procedure of bargaining. In life, often bargaining is less structured and the parties can make offers without structure, something which does not fit into the extensive game model.

The Bargaining Problem

For simplicity we will confine ourself to the case where there are just two bargainers, 1 and 2, who bargain on a partition of a desirable "pie" of size 1. The set of possible agreements will be $X = \{(x_1, x_2) | x_1 + x_2 = 1 \text{ and } x_i \geq 0\}$. The problem contains also details about "what happens if there is disagreement", denoted by D. In most of the lecture we assume that a player identifies the disagreement with the event of "getting nothing". In all models an agreement is reached by one of the players making an offer that the other accepts right away. An important assumption in all models is that players only care about the outcome of the bargaining and probably the time of the agreement but not on the path of offers and rejections which proceed it. This assumption rules out all sort of psychological considerations which one can easily imagine.

Model 1: "Take it or leave it"

Procedure: Player 1 makes an offer and player 2 accepts or rejects.
For the formal presentation of the situation as a game let H contain
▲ϕ, the initial history after which player 1 moves
▲(x) (for any $x \in X$) a history of length 1 after which player 2 moves

▲(x,Y), (for any $x \in X$), terminal histories where x is agreed
▲(x,N) (for any $x \in X$), where a disagreement emerges.
Payoffs Player i's payoff of a terminal history (x,Y) is taken to be x_i and player i's payoff of (x,N) is taken to be 0.

CLAIM 1: The game has a unique SPE in which player 1 makes the offer $e_1 = (1,0)$ and player 2 accepts all offers.

PROOF: Obviously the above is a SPE. Assume that (s_1,s_2) is a SPE.
Step 1 It must be that $s_2(x) = Y$ for any x with $x_2 > 0$
Step 2 $s_1(\phi) = e_1$ since if in equilibrium he offers x with $x_2 > 0$ he would do better by offering player 2 only $x_2/2$ (which is accepted as well).
Step 3 $s_2(e_1) = Y$ since otherwise player 1 does better by making any other offer.

Experiments (and common sense) show that most people do not behave according to the model's "prediction". More than it demonstrates the failure of SPE it draws our attention to the unrealistic assumptions regarding payoffs. First, people have preferences for fairness which lead many of us not to prefer the whole pie over what we perceive to be the fair partition. Second, many of the responders will feel insulted by a low offer and will prefer to reject an insulted offer rather than accepting it. No wonder, that many of the experimental outcomes are around the partitions $(1/2, 1/2)$ and $(2/3, 1/3)$. Both might be the offerer's best division given his preference containing the fairness consideration and subject to what player 2 will consider "unacceptable".

Model 2: Successive offers

Procedure: Assume now that one rejection is not the end of the process and bargaining can continue for T periods. , In period t player $i^t \in \{1,2\}$ has to make an offer and the other player $-i^t$ has to accept or reject it ($-i$ is the player who is not i).
For the formal presentation of the situation as a game let H contain:
▲$(x^1,N,x^2,N.\dots x^t,N)$ (for $0 \leq t < T$) after which player i^{t+1} has to move
▲$(x^1,N,x^2,N.\dots x^t)$ (for $1 \leq t < T$) after which player $-i^t$ has to move
▲$(x^1,N,x^2,N.\dots x^t,Y)$ (for $t \leq T$) a terminal history evaluated by player i as x_i^t.
▲$(x^1,N,x^2,N.\dots x^T,N)$ a terminal history evaluated by player i as 0.

CLAIM 2: In all SPE i^T 's payoff is 1 (and of the other 0).

PROOF: In any SPE:
Step 1: In ALL subgames after any $T-1$ offers were made and rejected, it must be that player i^T demands to himself the whole pie and the other accepts all offers (this follows from Claim 1).
Step 2: Player i^T 's payoff cannot be less than 1, since he can deviate to the strategy of rejecting all offers and always demands the whole pie to himself .
Step 3: Since the sum of payoffs is ≤ 1, $-i^T$'s payoff is 0.

Thus, in this model the last moment dominates the events in the bargaining. Whoever has the power to make the last move wins the pie.

Model 3: Infinite Successive offers

We have seen the important effect of the existence of terminal time point on the analysis of a game. For evaluating the importance further let us look at the case where only player 1 makes offers but, there is no finite time horizon, namely the players believe that after any rejection there will be another opportunity to agree.

Presenting this situation as an extensive game we construct an extensive game with the following set of possible histories:

▲ $(x^1, N, x^2, N \dots x^t, N)$ (for any $0 \leq t$) after which player 1 has to move

▲ $(x^1, N, x^2, N \dots x^t)$ (for any $1 \leq t$) after which player 2 has to move

▲ $(x^1, N, x^2, N \dots x^t, Y)$ (for $0 \leq t$) a terminal history evaluated by player i as x_i^t.

▲ $(x^1, N, x^2, N \dots x^t, N, \dots)$ infinite stream of offers and rejections, a terminal history evaluated by both players as 0.

Claim 3: Any partition $x^* \in X$ is an SPE outcome of this game.

Proof: Consider the strategies:

▲ Player 1: always offer x^*

▲ Player 2: always accepts an offer y with $y_2 \geq x_2^*$ and rejects all other offers.

Verify that this pair of strategies is a SPE.

There is also a SPE with no agreement:

The strategies can be described as follows (using automatae language).

■ Stage 1

▲ Player 1: always offer $(1, 0)$

▲ Player 2: rejects all offers

Transition to Stage 2 is done after player 1 makes an offer different than (1,0):

■ Stage 2

▲ Player 1: always offers $(0, 1)$

▲ Player 2: accepts only $(0, 1)$.

This equilibrium hints for a general way of constructing SPE. In order to deter player 1 from "becoming serious" the equilibrium suggests that the players interpret such a move is a "weakness" which leads to the equilibrium of the first type with $x^* = (0, 1)$.

Thus, with infinite horizon, the fact that player 1 is the only player to make offers does not give him any extra bargaining power.

Model 4: One sided offers, infinite horizon with discounting

Let us now modify Model 3 just in one point. We will introduce impatience to the model. Assume that the players evaluate a terminal history in which an agreement on x is reached at time t by, $x_i \delta_i^{t-1}$ where $\delta_i < 1$.

Note that the equilibrium scheme described in the previous section does not provide a SPE for this model (unless $x^* = (1, 0)$): If player 1 demands $x_1^* + \epsilon$, [for $\epsilon > 0$ small enough so that $x_2^* - \epsilon > \delta_2 x_2^*$ then player 2 prefers to say Y rather than following the strategy and receiving only x_2^* the next period.

Claim 4: Independently of the relative degrees of impatience, there is a unique SPE in which player 1 gets the whole pie.

Proof: Obviously for player 1 to always demand $(1, 0)$ and for player 2 to accept all offers is a SPE of this game.

Let M be the supremum of player 2's payoffs over all SPE of this game. Assume $M > 0$. For any SPE and in any subgame where player 2 has to respond to an offer, the continuation cannot yield player 2 strictly more than $\delta_2 M$; thus it must be that he plans to accept any offer which gives him $\delta_2 M + \epsilon$. Let u_2 be a SPE payoff for player 2 which is close to M. Player 1's payoff is at most $1 - u_2$. For u_2 close to M there is $\epsilon > 0$ so that $1 - (\delta_2 M + \epsilon) > 1 - u_2$ and thus player 1's can profitably deviate and by offering player 2's $\delta_2 M + \epsilon$ which is accepted. A contradiction to $M > 0$.

Thus, player 2's payoff in all SPE must be 0; therefore player 2 accepts all offers which give him strictly positive share of the unit. Player 1's payoff must be 1 because if there is a SPE where he gets $u < 1$ he can deviate and demand $(1+u)/2$ which will be accepted. Thus in any SPE player 1 offers $(1,0)$ and player 2 accepts all offers.

To summarize, with one sided offers and discounting, the only SPE gives the unit to the exclusive offerer, what shows the bargaining power of the privilege to make offers in this model. Note that this is true even if player 1 is very impatient!

EXAMPLE 5: **Alternating offers, infinite horizon with discounting**

Assume now that the players alternate offers; namely after an offer of player i is rejected, player j, at the next period, has to make an offer. The players' preferences in the game are derived from player i's utility function $U_i(x,t) = x_i^t \delta_i^{t-1}$ over the pairs (x,t).

The following are the possible histories:

▲ $(x^1,N,x^2,N....x^t,N)$ (for $t \geq 0$) after which player 1 moves if t is even and player 2 if t is odd.

▲ $(x^1,N,x^2,N....x^t)$ (for $t \geq 1$) after which player 1 or 2 has to move if t is even or odd accordingly.

▲ $(x^1,N,x^2,N....x^t,Y)$ (for $t \geq 0$) a terminal history evaluated by player i as $U_i(x^t,t)$.

▲ $(x^1,N,x^2,N....x^t,N,....)$ a terminal history evaluated by both players as 0.

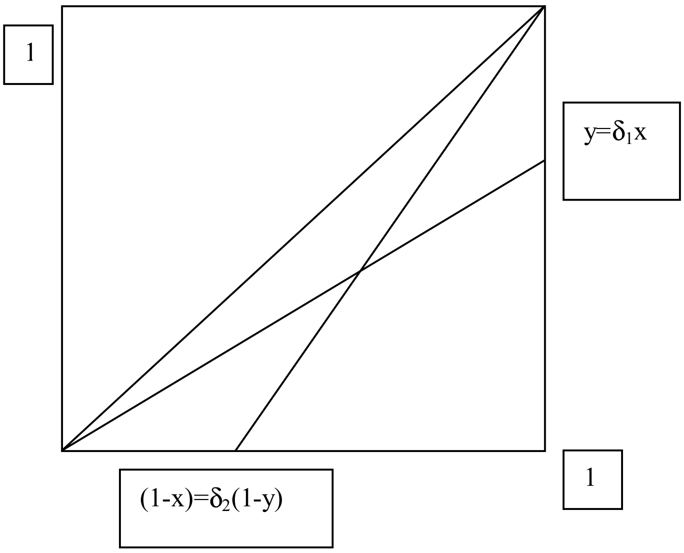
CLAIM 5: The model has a unique SPE in which

Player 1 always offers the partition $x^* = [(1-\delta_2)/(1-\delta_1\delta_2), \delta_2(1-\delta_1)/(1-\delta_1\delta_2)]$ and accepts any offer which gives him at least $\delta_1(1-\delta_2)/(1-\delta_1\delta_2)$.

Player 2 always demands $y^* = [\delta_1(1-\delta_2)/(1-\delta_1\delta_2), (1-\delta_1)/(1-\delta_1\delta_2)]$ and accepts any offer which gives him at least $\delta_2(1-\delta_1)/(1-\delta_1\delta_2)$.

PROOF: It is easy to check that the above is an SPE (note that the one deviation property has to be proved here!)

For proving uniqueness, it will be useful to look at the following diagram:



First notice the meaning of the above graphs. Each point on the axes is identified with a partition. The ray form the origin corresponds to the collection of $(x,y) \in X \times X$ for which player 1 is indifferent between x immediately and y at the next period. Similarly the other line is $\{(x,y) \in X \times X | (y,1) \sim_2 (x,0)\}$.

Let G_i be the game where i is the first offerer. Define M_i as the sup of player i's SPE payoffs in G_i. Let m_i be the corresponding infimum.

We show that $m_2 \geq 1 - \delta M_1$. Assume there is a SPE in G_2 in which Player 2's payoff $u_2 < 1 - \delta M_1$. Player 1 must accept any offer that give him more than δM_1 and thus player 2 can deviate and demand to himself an amount in the interval $(u_2, 1 - \delta M_1)$ which 1 accepts, a profitable deviation for 2.

Next we show that $1 - M_1 \geq \delta_2 m_2$ since assume that $1 < \delta_2 m_2 + M_1$. Take a SPE which gives player 1 almost M_1, in this equilibrium player 2 must get at least $\delta_2 m_2$ but the the sum of the players' payoffs cannot exceed 1.

From the two inequalities and the fact that $M_1 \geq x_1^*$ and $m_2 \leq y_2^*$ it follows that $M_1 = x_1^*$ and $m_2 = y_2^*$. For the analogous consideration it follows that $m_1 = x_1^*$ and $M_2 = y_2^*$.

Therefore all SPE yield a unique payoff for player 1 of x_1^*. In any SPE of G_1 player 2 can get at least $\delta_2 y_2^* = x_2^*$ (by rejecting the first offer) and since $x_1^* + x_2^* = 1$ it must be that in all SPE his payoff is exactly x_2^*. This means that in all SPE agreement is reached at the first period. This means that in all SPE player 1 offers x^* and player 2 always offers y^*; Player 1 accepts offers which are at least good for him as y^* and player 2 accepts offers which are at least as good for him as x^* only.

DISCUSSION: Efficiency; Comparative statics

Problem set 16

1. **(Easy Exercise)** Formulate and analyze a multi stage bargaining model where two (impatient) parties submit their proposals simultaneously and an agreement is reached only if these proposals match.

2. (**Exercise)** Analyze the alternating offers model under the assumption that each player i bears a fixed bargaining cost of c_i and $c_1 < c_2$.

3. (**Exercise)** Consider the alternating offers model with the set of agreements $X = [0,1]$ and each player i holds a utility function $U_i(x,t) = u_i(x)\delta_i^t$ where u_1 is an increasing and u_2 is a decreasing function and $u_1(0) = u_2(1) = 0$ and $u_i(D) = 0$. Assume that there is a unique pair x^* and y^* such that $\delta_1 u_1(x^*) = u_1(y^*)$ and $\delta_2 u_2(y^*) = u_2(x^*)$.

Show that there is an SPE with x^* is agreed right away. Show that this is the unique SPE of the game.

4. **(Exercise)** Show that if the function $U_i(x,t) = u_i(x)\delta_i^{t-1}$ represents player i's preference over the space $X \times T$ then for *any* $\delta < 1$ there is a function $v_i(x)$ so that $v_i(x)\delta^{t-1}$ represents player i's time preference as well.

5. (**Exercise)** Show that if the two players in Problem 3 have time preferences represented by $U_i(x,t) = v_i(x)\delta^{t-1}$ (the same discount rate!) then the SPE outcome $x^*(\delta)$ converges, when $\delta \to 1$, to $argmax_{x \in X}\, u_1(x)u_2(x)$.

6. **(Exercise)** Modify the alternating offers model 5 with the addition that each player can opt out at the end of each period t and forces an outcome which is evaluated by player i as $d_i^*\delta_i^{t-1}$ Assume that for both i, d_i^* is less than the payoff from the unique SPE of the game without outside options. Show that the existence of outside options in this model does not change the set of SPE.

7. (**Thinking)** Recently there is much interest in hyperbolic time preference relations of the type $U_i(x,1) = v_i(x)$ and $U_i(x,t) = v_i(x)\beta\delta^{t-1}$ for $t > 1$. What changes are needed in the analysis of the alternating offers model to cop with such preference relations?