Lecture 12: Zero Sum Games
Readings: Osborne and Rubinstein Ch 2.5

Strictly Competitive Games

Only in limited classes of games can we say something about the qualitative character of the equilibria. One such class of games is that in which there are two players, whose preferences are diametrically opposed. For convenience assume $N = \{1,2\}$.

A strategic game $\langle\{1,2\},(A_i),(\succsim_i)\rangle$ is *strictly competitive* if for any $a \in A$ and $b \in A$ we have $a \succsim_1 b$ if and only if $b \succsim_2 a$.

A strictly competitive game is sometimes called *zero-sum* because if player 1's preference relation $\succsim_1$ is represented by the payoff function u_1 then player 2's preference relation is represented by $u_2 = -u_1$.

We identify a pattern of strategic reasoning of a special kind. We say that player i *maxminimizes* if he chooses an action that is best for him under the assumption that whatever he does, player j will choose his action to hurt him as much as possible.

We interpret it in two possible ways. (1) A decision making method: the player always assume the worst and try to minimize the disaster. (2) A strategic reasoning: in spite of the simultaneousness, a player anticipates that his opponent will respond optimally (from the opponent's point of view).

Main message: We will show that a strictly competitive game possesses a Nash equilibrium, a pair of actions is a Nash equilibrium if and only if the action of each player is a maxminimizer.

This provides a link between individual decision-making and the reasoning behind the notion of Nash equilibrium. It will follow that for strictly competitive games that possess Nash equilibria all equilibria yield the same payoffs.

Definition: Let $\langle\{1,2\},(A_i),(u_i)\rangle$ be a strictly competitive strategic game. The action $z^* \in A_1$ is a maxminimizer for player 1 if $\min_{y\in A_2} u_1(z^*,y) \geq \min_{y\in A_2} u_1(x,y)\ \forall x \in A_1$. That is, a maxminimizer for player i is an action that maximizes the payoff that player i can *guarantee*.

Let $\succsim_i$ be represented by a payoff function u_i. Without loss of generality, assume that $u_2 = -u_1$.

Lemma The maxminimization of player 2's payoff is equivalent to the minmaximization of player 1's payoff. That is, let $\langle\{1,2\},(A_i),(u_i)\rangle$ be a strictly competitive strategic game.
(a) $\max_{y\in A_2} \min_{x\in A_1} u_2(x,y) = -\min_{y\in A_2} \max_{x\in A_1} u_1(x,y)$.
(b) $y \in A_2$ solves the problem $\max_{y\in A_2} \min_{x\in A_1} u_2(x,y)$ iff it solves the problem $\min_{y\in A_2} \max_{x\in A_1} u_1(x,y)$.

Proof Note that for any function f we have $\min_z(-f(z)) = -\max_z f(z)$ and $\arg\min_z(-f(z)) = \arg\max_z f(z)$.
Thus, for every $y \in A_2$ $-\min_{x\in A_1} u_2(x,y) = \max_{x\in A_1}(-u_2(x,y)) = \max_{x\in A_1} u_1(x,y)$.
$\max_{y\in A_2} \min_{x\in A_1} u_2(x,y) = -\min_{y\in A_2}[-\min_{x\in A_1} u_2(x,y)] = -\min_{y\in A_2} \max_{x\in A_1} u_1(x,y)$;
in addition $y \in A_2$ is a solution of the problem $\max_{y\in A_2} \min_{x\in A_1} u_2(x,y)$ if and only if it is a solution of the problem $\min_{y\in A_2} \max_{x\in A_1} u_1(x,y)$.

Proposition Let $G = \langle\{1,2\},(A_i),(u_i)\rangle$ be a strictly competitive strategic game.
(a) If (x^*,y^*) is a Nash equilibrium of G then x^* is a maxminimizer for player 1 and y^* is a maxminimizer for player 2.
(b) If (x^*,y^*) is a Nash equilibrium of G then
$\max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y) = u_1(x^*,y^*)$, and thus all Nash equilibria of G yield the same payoffs.
(c) If $\max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y)$ (and thus, in particular, if G has a Nash equilibrium (see part b)), x^* is a maxminimizer for player 1, and y^* is a maxminimizer for player 2, then (x^*,y^*) is a Nash equilibrium of G. proposition

Proof (a) and (b).
Let (x^*,y^*) be a Nash equilibrium of G.
Then $u_2(x^*,y) \leq u_2(x^*,y^*)$ for all $y \in A_2$ or, since $u_2 = -u_1$, $u_1(x^*,y^*) \leq u_1(x^*,y)$ for all $y \in A_2$.
Hence $\min_y u_1(x^*,y) = u_1(x^*,y^*)$
For any $x \in A_1$ we have $\min_y u_1(x,y) \leq u_1(x,y^*)$.
Since (x^*,y^*) be a Nash equilibrium of G we have $u_1(x,y^*) \leq u_1(x^*,y^*)$ for all $x \in A_1$. Thus $u_1(x^*,y^*) = \max_x \min_y u_1(x,y)$ and x^* is a maxminimizer for player 1.
An analogous argument for player 2 establishes that y^* is a maxminimizer for player 2 and $u_2(x^*,y^*) = \max_y \min_x u_2(x,y)$.
By the Lemma $u_1(x^*,y^*) = -u_2(x^*,y^*) = -\max_y \min_x u_2(x,y) = \min_y \max_x u_1(x,y)$.

Proof of (c):
Let $v^* = \max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y)$.
By the Lemma we have $\max_y \min_x u_2(x,y) = -v^*$.
Since x^* is a maxminimizer for player 1 we have $u_1(x^*,y) \geq v^*$ for all $y \in A_2$;
Since y^* is a maxminimizer for player 2 we have $u_2(x,y^*) \geq -v^*$ and thus $u_1(x,y^*) \leq v^*$ for all $x \in A_1$.
Letting $y = y^*$ and $x = x^*$ in these two inequalities we obtain $u_1(x^*,y^*) = v^*$
Using the fact that $u_2(x^*,y^*) = -u_1(x^*,y^*)$, we conclude that (x^*,y^*) is a Nash equilibrium of G.

▸ By (c) a Nash equilibrium can be found by solving the problem $\max_x \min_y u_1(x,y)$.
▸ By (a) and (c) Nash equilibria of a strictly competitive game are *interchangeable*: if (x,y) and (x',y') are equilibria then so are (x,y') and (x',y).
▸ Always $\max_x \min_y u_1(x,y) \leq \min_y \max_x u_1(x,y)$
since $u_1(x',y) \leq \max_x u_1(x,y)$ for all y,
and thus $\min_y u_1(x',y) \leq \min_y \max_x u_1(x,y)$ for all x.
▸In *Matching Pennies*, $\max_x \min_y u_1(x,y) = -1 < \min_y \max_x u_1(x,y) = 1$.
▸(b) shows that $\max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y)$ for any 0-sum game that has NE.

If $\max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y)$ then we say that this payoff, the equilibrium payoff of player 1, is the *value of the game*.

Problem set 12

1. **(Exercise)** Let G be a strictly competitive game that has a Nash equilibrium.

▲ Show that if some of player 1's payoffs in G are increased in such a way that the resulting game G' is strictly competitive then G' has no equilibrium in which player 1 is worse off than she was in an equilibrium of G. (Note that G' may have no equilibrium at all.)

▲ Show that the game that results if player 1 is prohibited from using one of her actions in G does not have an equilibrium in which player 1's payoff is higher than it is in an equilibrium of G.

▲ Give examples to show that neither of the above properties necessarily holds for a game that is not strictly competitive.

2. **(Exercise)** Formulate a formal concept which will capture the situation that in a zero-sum game where each player has to choose an action from a set X, player 1 is discriminated in favor. What can you say about the Nash equilibrium in such a game (assuming it exists)?