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Lecture 11: Introduction to Game Theory
Readings: Osborne and Rubinstein Ch1 Ch2.1-4

We view Game theory as a bag of tools designed to help us understand the phenomena that we observe when decision-makers interact. The basic assumptions that underlie the theory are that decision-makers pursue well-defined exogenous objectives (they are *rational*) and take into account their knowledge or expectations of *other* decision-makers' behavior (they *reason strategically*). The models of game theory are highly abstract representations of classes of real-life situations. Their abstractness allows them to be used to study a wide range of phenomena but has the disadvantage of missing many of the real life considerations which effect players' behavior.

A discussion of the word "game".

Game Theory and the Theory of Competitive Equilibrium

Contrast GT with the theory of competitive equilibrium. Game theoretic reasoning takes into account the attempts by each decision-maker to obtain, prior to making his decision, information about the other players' behavior, while competitive reasoning assumes that each agent is interested only in some environmental parameters, such as prices, even though these parameters are determined by the actions of all agents. For example, when considering picking a gate to enter a stadium, a competitive equilibrium approach will suggest that an agent responses to the "waiting time" (which is effected by the agents decisions) whereas a game theoretic analysis of the situation requires that each agent's action be optimal given his anticipation of all other agents' actions.

Games and Solutions

A game is a description of strategic interaction that includes the constraints on the actions that the players *can* take and the players' interests, but does not specify the actions that the players *do* take. A *solution* is a systematic description of the outcomes that may emerge in a family of games. A solution is a "method of reasoning about the situation". Game theory suggests reasonable solutions for classes of games and examines their properties.

Noncooperative and Cooperative Games

In all game theoretic models the basic entity is a *player*. A player may be interpreted as an individual or as a group of individuals making a decision. Once we define the set of players, we may distinguish between two types of models:
"noncooperative": those in which the sets of possible actions of *individual* players are primitives
"cooperative" those in which the sets of possible joint actions of *groups* of players are primitives.

The Steady State and Deductive Approaches

There are two conflicting interpretations of solutions for games. The *steady state* interpretation is closely related to that which is standard in economics. Game theory, like other sciences, deals with regularities. The steady state interpretation treats a game as a model designed to explain some regularity observed in a family of similar situations. Each participant "knows" the equilibrium and tests the optimality of his behavior given this knowledge, which he has acquired from his long experience. The *deductive* interpretation, by contrast, treats a game in isolation, as a "one-shot" event, and attempts to infer the restrictions that rationality imposes on the outcome; it assumes that each player deduces how the other players will behave simply from principles of rationality. The literature is often confused between the two interpretations.

Bounded Rationality

When we talk in real life about games we often focus on the asymmetry between individuals in their abilities. For example, some players may have a clearer perception of a situation or have a greater ability to analyze it. These differences, which are so critical in life, are missing from game theory in its current form. Modeling asymmetries in abilities and in perceptions of a situation by different players is a fascinating challenge for future research, which models of "bounded rationality" have begun to tackle.

Strategic games

A strategic game is a model of interactive decision-making in which each decision-maker chooses his plan of action once and for all, and these choices are made simultaneously. The model consists of

a finite set N (the set of *players*)

for each player $i \in N$ a non-empty set A_i (the set of *actions* available to player i)

for each player $i \in N$ a preference relation $\succsim_i$ on $A = \times_{j \in N} A_j$ (the *preferences* of player i).

The requirement that the preferences of each player i be defined over A, rather than A_i, distinguishes a strategic game from a decision problem: each player may care not only about his own action but also about the actions taken by the other players.

A player may be an individual human being or any other decision-making entity like a government, a board of directors, an agent in a particular date, or even a flower or an animal.

The model places no restrictions on the set of actions available to a player. A player's preference relation may simply reflect the player's feelings about the possible outcomes or, in the case of an organism that does not act consciously, the chances of its reproductive success.

However, the range of application of the model is limited by the requirement that we associate with each player a preference relation.

When the preference relation of player i can be represented by a utility function $u_i\colon A \to \Re$ we refer to it as a payoff function. In such a case we denote the game by $\langle N, (A_i), (u_i) \rangle$ rather than $\langle N, (A_i), (\succsim_i) \rangle$. A two player finite strategic game can be described conveniently by a *bi matrix*.

In some situations the players' preferences are most naturally defined not over action profiles but over their consequences. Let C be a set of *consequences*, and let $g\colon A \to C$ associates a consequence with each outcome. Let $(\succsim_i^*)$ be a profile of preference relations over C. Then the preference relation $\succsim_i$ of each player i in the strategic game is defined as follows: $a \succsim_i b$ if and only if $g(a) \succsim_i^* g(b)$.

Comments on Interpretation

A common interpretation of a strategic game is that it is a model of an event that occurs only once; each player knows the details of the game and the fact that all the players are "rational" , and the players choose their actions simultaneously and independently. The main picture we have in mind is of players sitting in front of terminals in different locations. First the players' possible actions and payoffs are described publicly. Then each player chooses an action by sending a message to a central computer; the players are informed of their payoffs when all the messages have been received. However, the model has a wider applicability. For a situation to be modeled as a strategic game it is important only that the players make decisions independently, no player being informed of the choice of any other player prior to making his own decision. Each player is unaware, when choosing his action, of the choices being made by the other players; there is no information (except the primitives of the model) on which a player can base his expectation of the other players' behavior.

Sometimes, we think about a strategic game as an instance which occurs after the players had a long experience about playing the game and they hold expectations of the other players' behavior on the basis of information about the way that the game or a similar game was played in the past. When the game captures a reoccurring situation we could think about it as a strategic game only if there are no strategic links between the plays. That is, an individual who plays the game many times is concerned only with his instantaneous payoff and ignore the effects of his current action on the other players' future behavior. The model of a repeated game deals with series of strategic interactions in which such inter-temporal links *do* exist.

Nash Equilibrium

The most commonly used solution concept in game theory is that of Nash equilibrium. This notion captures a *steady state* of the play of a strategic game in which each player holds the correct expectation about the other players' behavior and acts rationally. It does not attempt to examine the process by which a steady state is reached.

In Nash equilibrium no player can profitably deviate, given the actions of the other players.

Definition A Nash equilibrium of a strategic game $\langle N, (A_i), (\succsim_i)\rangle$ is a profile $a^* \in A$ of actions with the property that for every player $i \in N$ we have

$$(a^*_{-i}, a^*_i) \succsim_i (a^*_{-i}, a_i) \;\; \forall a_i \in A_i.$$

For any $a_{-i} \in A_{-i}$ define $B_i(a_{-i})$ to be the set of player i's best actions given a_{-i}: $B_i(a_{-i}) = \{a_i \in A_i : (a_{-i}, a_i) \succsim_i (a_{-i}, a'_i)$ for all $a'_i \in A_i\}$. We call B_i the best-response function of player *i*. A Nash equilibrium is a profile a^* of actions for which $a^*_i \in B_i(a^*_{-i})$ for all $i \in N$. This formulation of the definition points us to a (not necessarily efficient) method of finding Nash equilibria: first calculate the best response function of each player, then find a profile a^* of actions for which $a^*_i \in B_i(a^*_{-i})$ for all $i \in N$. If the functions B_i are singleton-valued then the second step entails solving $|N|$ equations in the $|N|$ unknowns $(a^*_i)_{i \in N}$.

Note that for the degenerate case of a single player Nash equilibrium is equivalent to optimizing the preference relation.

Interpretation of Nash Equilibrium

Emphasize the idea that this information which can be added to the model without being conflicting with the rationality of the individuals.

Examples

The following games are very simple: in each game there are just two players and each player has only two possible actions. Each game captures the essence of a type of common strategic interactions.

Example (*Battle of the sexes(BoS)*)

BoS models a situation in which players wish to coordinate their behavior, but have conflicting interests.

he/she	*M*	*F*
M	2, 1	0, 0
F	0, 0	1, 2

The game has two Nash equilibria: That is, there are two steady states: one in which both players always choose *M* and one in which they always choose *M*.

Example (*A coordination game*) Two people wish to go out together, but in this case they agree on the more desirable activity. In contrast to BoS, the players have a mutual interest in reaching one outcome.

he/she	*M*	*F*
M	2, 2	0, 0
F	0, 0	1, 1

Like BoS, the game has two Nash equilibria: . The notion of Nash equilibrium does not rule out a steady state in which the outcome is the inferior equilibrium .

Example (The *Prisoner's Dilemma*) This is the most and much over-discussed game. Two suspects in a crime are put into separate cells. If they both confess, each will be sentenced to three years in prison. If only one of them confesses, he will be freed and used as a witness against the other, who will receive a sentence of four years. If neither confesses, they will both be convicted of a minor offense and spend one year in prison. Choosing a convenient payoff representation for the preferences, we have the game in

he/she	*Coop*	*Defect*
Coop	3, 3	0, 4
Defect	4, 0	1, 1

This is a game in which there are gains from cooperation—the best outcome for the players is that both cooperate—but each player has an incentive to be a "free rider". Note however, that the gains are to the ... prisoners. Thus, the question whether this is a "problem" depends on our ethical assessment of the situation.

Example (*Hawk–Dove*) Two animals are fighting over some prey. Each can behave like a dove or like a hawk. The best outcome for each animal is that in which it acts like a hawk while the other acts like a dove; the worst outcome is that in which both animals act like hawks. Each animal prefers to be hawkish if its opponent is dovish and dovish if its opponent is hawkish.

he/she	*Dove*	*Hawk*
Dove	3,3	1,4
Hawk	4,1	0,0

The game has two Nash equilibria, , corresponding to two different conventions about the player who yields.

Example (*Matching Pennies*) Each of two people chooses either Head or Tail. If the choices differ, person 1 pays person 2 a dollar; if they are the same, person 2 pays person 1 a dollar. Each person cares only about the amount of money that he receives.

	Head	*Tail*
Head	1,−1	−1,1
Tail	−1,1	1,−1

Such a game, in which the interests of the players are diametrically opposed, is called "strictly competitive". The game *Matching Pennies* has no Nash equilibrium.

Compare the game with "odd or even": In this game $N = \{odd, even\}$ $A_i = \{1,2,3,4,5\}$ (the # of "fingers") and player *odd* wins iff $a_1 + a_2$ is odd. If the players recognize that all odd strategies and all even strategies are "the same" then the game is transformed into the matching pennies. But, does the fact that there are more odd strategies improve the chances that the even will win?

Existence of a Nash Equilibrium

The conditions under which the set of Nash equilibria of a game is non-empty have been investigated extensively. The main purpose of an existence result is that it shows that the model is *consistent* with a steady state solution. The existence of equilibria for a family of games allows us to study properties of these equilibria without finding them explicitly and without taking the risk that we are studying the empty set.

To show that a game has a Nash equilibrium it suffices to show that there is a profile a^* of actions such that $a_i^* \in B_i(a_{-i}^*)$ for all $i \in N$. Define the set-valued function $B: A \to A$ by $B(a) = \times_{i \in N} B_i(a_{-i})$. Then a Nash equilibrium can be written as $a^* \in B(a^*)$. *Fixed point theorems* give conditions on B under which there indeed exists a^* for which $a^* \in B(a^*)$.

Lemma[Kakutani's fixed point theorem] Let X be a compact convex subset of $\Re^n$ and let $f: X \to X$ be a set-valued function for which

▲for all $x \in X$ the set $f(x)$ is non-empty and convex

▲the graph of f is closed (i.e. for all sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n \in f(x_n)$ for all n, $x_n \to x$, and $y_n \to y$, we have $y \in f(x)$).

Then there exists $x^* \in X$ such that $x^* \in f(x^*)$.

Each of the following four conditions is necessary for Kakutani's theorem.

(i) X is compact ($f(x) = \{x^2\}$ on $(0,1)$).

(ii) X is convex (rotate the circle).

(iii) $f(x)$ is convex for each $x \in X$. ($f(x) = \{0\}$ for all $x < 1/2$, $f(x) = \{1\}$ for all $x > 1/2$ and $f(1/2) = \{0,1\}$ on $[0,1]$.

(iv) f has a closed graph ($f(x) = \{1 - x\}$ for all $x \neq 1/2$ and $f(1/2) = \{0\}$.

Proposition The strategic game $\langle N, (A_i)(\succsim_i)\rangle$ has a Nash equilibrium whenever for all $i \in N$ the set A_i is a non-empty compact convex subset of a Euclidian space and the preference relation $\succsim_i$ is continuous quasi-concave on A_i.

Proof. Define $B: A \to A$ by $B(a) = \times_{i \in N} B_i(a_{-i})$ (where B_i is the best-response function of player i. For every $i \in N$ the set $B_i(a_{-i})$ is non-empty since $\succsim_i$ is continuous and A_i is compact, and is convex since $\succsim_i$ is quasi-concave on A_i; B has a closed graph since each $\succsim_i$ is continuous (explain!). Thus by Kakutani's theorem B has a fixed point which is a Nash equilibrium of the game.

Problem set 11

1. (**Exercise) First price auction** An object is to be assigned to a player in the set $\{1,\ldots,n\}$ in exchange for a payment. Player i's valuation of the object is v_i, and $v_1 > v_2 > \cdots > v_n > 0$. The players simultaneously submit sealed bids (nonnegative numbers), and the object is given to the player with the lowest index among those who submit the highest bid, in exchange for a payment. In a *first price* auction the payment that the winner makes is the price that he bids.

Formulate the first price auction as a strategic game and analyze its Nash equilibria. In particular, show that in all equilibria player the buyer with the highest reservation value obtains the object.

2**. (Exercise)** S**econd Price Auction** In a *second price auction* the payment that the winner makes is the highest bid among those submitted by the players who do not win (so that if only one player submits the highest bid then the price paid is the *second* highest bid).

Show that in a second price auction the bid v_i of any player i is a *weakly dominant* action: player i's payoff when he bids v_i is at least as high as his payoff when he submits any other bid, regardless of the actions of the other players. Show that nevertheless there are ("inefficient") equilibria in which the winner is not player 1.

3. **Exercise** (**A War of Attrition**) Two players are involved in a dispute over an object. The value of the object to player i is $v_i > 0$. Time is modeled as a continuous variable that starts at 0 and runs indefinitely. Each player chooses when to concede the object to the other player; if the first player to concede does so at time t, the other player obtains the object at that time. If both players concede simultaneously, the object is split equally between them, player i receiving a payoff of $v_i/2$. Time is valuable: until the first concession each player loses one unit of payoff per unit of time.

Formulate the war of attrition as a strategic game and show that in all Nash equilibria one of the players concedes immediately.

4. **Exercise** (**A Location Game**) Each of n people chooses whether or not to become a political candidate, and if so which position to take. There is a continuum of citizens, each of whom has a favorite position; the distribution of favorite positions is given by a density function f on $[0,1]$ with $f(x) > 0$ for all $x \in [0,1]$. A candidate attracts the votes of those citizens whose favorite positions are closer to his position than to the position of any other candidate; if k candidates choose the same position then each receives the fraction $1/k$ of the votes that the position attracts. The winner of the competition is the candidate who receives the most votes. Each person prefers to be the unique winning candidate than to tie for first place, prefers to tie for first place than to stay out of the competition, and prefers to stay out of the competition than to enter and lose.

Formulate this location situation as a strategic game, find the set of Nash equilibria when $n = 2$, and show that there is no Nash equilibrium when $n = 3$.

5. (**Symmetric games**) Consider a two-person strategic game that satisfies the

conditions of the existence theorem. Let $N = \{1,2\}$ and assume that the game is *symmetric*: $A_1 = A_2$ and $(a_1,a_2) \succsim_1 (b_1,b_2)$ if and only if $(a_2,a_1) \succsim_2 (b_2,b_1)$ for all $a \in A$ and $b \in A$. Use Kakutani's theorem to prove that there is an action $a_1^* \in A_1$ such that (a_1^*,a_1^*) is a Nash equilibrium of the game. (Such an equilibrium is called a *symmetric equilibrium*.) Give an example of a finite symmetric game that has only asymmetric equilibria.