# Solution for problem set 9

## 1.

Consider the strategy of player 1 in which she chooses $C$ then $D$, followed by $C$ and two $D$'s, followed by $C$ and three $D$'s, and so on, independently of the other players' behavior. Since there is no cycle in this sequence, the strategy cannot be executed by a machine with finitely many states.

## 2.

Consider a two-player game in which the constituent game has two payoff profiles, (1,0) and (0,1). Let $(v^t)$ be the sequence of payoff profiles of the constituent game in which $v^1 = (0,1)$ and $v^t = (1,0)$ for all $t \geq 2$. The payoff profile associated with this sequence is $(\delta_1, 1-\delta_2)$. Whenever $\delta_1 \neq \delta_2$ this payoff profile is not feasible. In particular, when $\delta_1$ is close to 1 and $\delta_2$ is close to 0 the payoff profile is close to (1, 1), which Pareto dominates all feasible payoff profiles of the constituent game.

## 3.

Let player 2's machine be $\langle Q_2, q_2^0, f_2, \tau_2\rangle$; a machine that induces a payoff for player 1 of at least $v_1$ is $\langle Q_1, q_1^0, f_1, \tau_1\rangle$ where

- $Q_1 = Q_2$.
- $q_1^0 = q_2^0$.
- $f_1(q) = b_1(f_2(q))$ for all $q \in Q_2$.
- $\tau_1(q, a) = \tau_2(q, a)$ for all $q \in Q_2$ and $a \in A$.

This machine keeps track of player 2's state and always responds to player 2's action in such a way that it obtains a payoff of at least $v_1$.

**4.**

Let $a^*$ be sequence of outcomes of length k in which every player obtains on average more than his payoff in the inferior NE of G. A strategy profile in the T-period repeated game that generates a sequence of outcomes for which the average payoff profile is close to $u(a^*)$ when T is large has the following form. Define two stages of the game, the first of length T-R and the second of length R (T and R will be determined latter).

In stage 1 start the game by playing $a_i^*$. After every cycle of length k if no single player has deviated from $a^*$ and t<T-R-k+1 then continue playing $a^*$, if t=T-R+1 move to play the superior NE. Otherwise move to play the inferior NE for the rest of the game.

First note that there is no need to check if any player will want to deviate during the last R periods of the game or when the players play the inferior NE since these are NE of the constituent game.

We will now determine R so no player will profit from deviating from $a^*$. It is sufficient to ensure that no player will want to deviate during the last cycle of $a^*$ since $u(a^*)$ is higher than the payoff of the inferior NE. Let M be the highest payoff that any player can receive in the constituent from all outcomes.

Take R to be the lowest integer so that the following inequality holds for all *i*.

(*) $kM + Ru_i(\text{inferior NE}) \le ku_i(a^*) + Ru_i(\text{superior NE})$. Such an R exists since $u_i(\text{superior NE}) - u_i(\text{inferior NE}) > 0$ for all *i*. Condition (*) ensures that even if player *i* deviates from $a^*$ for the whole k periods and receives the maximal payoff during these periods he will not profit from deviating since latter on the inferior NE will be played for R periods instead of the superior NE.

Now we will determine T that guarantees that the average payoff profile of $G^T$ is close to $u(a^*)$. Notice that T does not depend on R.

Take T to be large enough so that $\left|\frac{1}{T}\left(u_i(a^*)(T-R)+u_i(\text{superior NE})R\right)-u_i(a^*)\right|<\varepsilon$ for all $i$.

A sketch of the solution of the original question that any strictly enforceable payoff profile can be achieved in a SPE of $G^T$ is also presented.
The solution to this question draws upon ideas in the proofs of the "perfect folk theorem for the discounting criterion" and of the "Nash folk theorem for finitely repeated games" (propositions 151.1 and 156.1 in the text book respectively). For the proof to be accurate (and for the claim to be correct) one has to rely on the assumptions made in the "perfect folk theorem for the discounting criterion". Basically, assume that there is a collection $(a(i))_{i\in N}$ of strictly enforceable outcomes of G such that for every player $i \in N$ we have $a^* \succ_i a(i)$ and $a(j) \succ_i a(i)$ for all $j \in N \setminus \{i\}$.

Let $a^*$ be a strictly enforceable outcome of G. A strategy profile in the T-period repeated game that generates a sequence of outcomes for which the average payoff profile is close to $u(a^*)$ when T is large has the following form. There are 3 stages. Throughout the first 2 stages each player $i$ chooses $a_i^*$ so long as no player deviates. In the third stage the players adhere, in the absence of a deviation, to a sequence of Nash equilibria of the constituent game for which each player's average payoff exceeds his lowest Nash equilibrium payoff in the constituent game. Deviations are punished as follows. A deviation that occurs during the first stage is punished by the other players' using an action that holds the deviant to his minmax payoff for long enough to wipe out his gain. After this punishment is complete, a state of "reconciliation" is entered for long enough to reward the players who took part in the punishment for completing their assignment. A deviation by some player $i$ that occurs during the second stage is ignored until the beginning of the 3rd stage, during which the worst Nash equilibrium for player $i$ is executed in every period. Deviations during the last stage do not need to be punished since the outcome in every period is Nash equilibrium of the constituent game. The length of the 2nd stage is chosen to be large enough that for a player who deviates in the last period of the first stage both the punishment and the subsequent reconciliation can be completed during the second stage. Given the length of the 2nd stage, the length of the 3rd stage is chosen to be large enough that a player who deviates in the first period of the second stage is worse off given his punishment, which begins in the first period of the third stage. The lower bounds on the lengths of the second and third stages are independent of T, so that for T large enough the average payoff profile induced by the strategy profile is close to $u(a^*)$.

## 5.

First we bring the solution of the easier case where there is one NE where the payoff exceeds the minmax payoff of all players. The solution is from page 156 in the text book proposition 156.1.

Consider the strategy of player *i* that is carried out by the following machine. The set of states consists of $Norm^t$ for $t=1,\ldots,T-L$ ($L$ is determined later), *Nash* and *P(j)* for each $j \in N$. Each player *i* chooses $a_i^*$ in $Norm^t$ for all values of *t*, $b_i$ in *Nash* and punishes player *j* by choosing $(p_{-j})_i$ in *P(j)*. If a single player *j* deviates in state $Norm^t$, then there is a transition to *P(j)*; otherwise there is a transition to $Norm^{t+1}$ if $t<T-L$ and to *Nash* if $t=T-L$. Once reached, the states *P(j)* and *Nash* are never left. The outcome is that $a^*$ is chosen in the first *T-L* periods and *b* is chosen is the last *L* periods. To summarize, player *i*'s machine is the following.

- Set of states $\{Norm^t : 1 \le t \le T-L\} \cup \{P(j) : j \in N\} \cup \{Nash\}$
- Initial state: $Norm^1$
- Output function: In $Norm^t$ choose $a_i^*$, in *P(j)* choose $(p_{-j})_i$, and in Nash choose $b_i$ .

- Transition function: from $Norm^t$ move to $Norm^{t+1}$ unless $t=T-L$, in which case move to *Nash*, or exactly one player, say $j$, deviated from $a^*$, in which move to $P(j)$. $P(j)$ for any $j \in N$ and *Nash* are absorbing.

It remains to specify $L$. A profitable deviation is possible only in one of the $Norm^t$. To deter such a deviation we require L to be large enough that $\max_{a_i \in A_i} u_i(a^*_{-i}, a_i) - u_i(a^*) \leq L(u_i(Nash) - v_i)$ for all $i \in N$. Finally, in order to obtain a payoff within $\varepsilon$ of $u_i(a^*)$ we choose T* so that

$$\left| \frac{1}{T^*}\left[(T^* - L)u_i(a^*) + Lu_i(Nash)\right] - u_i(a^*) \right| < \varepsilon \text{ for all } i \in N.$$

For the more complicated case:
For each $i \in N$ let $a'^i$ be a Nash equilibrium of $G$ in which player $i$'s payoff exceeds his minmax payoff $v_i$. To cover this case, the strategy in the proof of Proposition 156.1 needs to be modified as follows.

- The single state *Nash* is replaced by a collection of states $Nash^i$ for $i \in N$.
- In $Nash^i$ each player $j$ chooses the action $a'^i_j$.
- The transition from $Norm^{T-L}$ is to $Nash^1$, and the transition from $Nash^k$ is to $Nash^{k+1(\text{mod } |N|)}$
- $L = K|N|$ for some integer $K$ and $K$ is chosen to be large enough that $\max_{a_i \in A_i} u_i(a^*_{-i}, a_i) - u_i(a^*) \leq K(\sum_{j \in N} u_i(a'^j) - |N| v_i)$ for all $i \in N$.

$T^*$ is chosen so that $|[(T^* - L)u_i(a^*) + K \sum_{j \in N} u_i(a'^j)]/T^* - u_i(a^*)| < \varepsilon$.

It is straightforward to see that this is not a SPE; for some $j,i$ there is a player $i$ that will be better off by deviating from the described strategy in state $P(j)$, unless for each $j \in N$ we have that $P(j)$ is also a Nash equilibrium of the constituent game.

## 6. (Long- and short-lived players)

First note that in any subgame perfect equilibrium of the game, the action taken by the opponent of player 1 in any period $t$ is a one-shot best response to player 1's action in period $t$.

*a*. The game has a unique subgame perfect equilibrium, in which player 1 chooses $D$ in every period and each of the other players chooses $D$.

*b*. Choose a sequence of outcomes $(C, C)$ and $(D, D)$ whose average payoff to player 1 is $x$. Player 1's strategy makes choices consistent with this path so long as the previous outcomes were consistent with the path; subsequent to any deviation it chooses $D$ for ever. Her opponent's strategy in any period $t$ makes the choice consistent with the path so long as the previous outcomes were consistent with the path, and otherwise chooses $D$.

## 7. (Example with discounting)

We have $(v_1,v_2)=(1,1)$, so that the payoff of player 1 in every subgame perfect equilibrium is at least 1. Since player 2's payoff always exceeds player 1's payoff we conclude that player 2's payoff in any subgame perfect equilibria exceeds 1. The path $((A, A), (A, A), \ldots)$ is not a subgame perfect equilibrium outcome path since player 2 can deviate to $D$, achieving a payoff of 5 in the first period and more than 1 in the subsequent subgame, which is better for him than the constant sequence $(3, 3, \ldots)$.

*Comment* We use only the fact that player 2's discount factor is at most 1/2.