Solution for Problem set 8

1.

Consider the next multi stage bargaining model: In every stage, the two (impatient) parties submit their proposals simultaneously. If the proposals match, an agreement is reached. Note that there might be more than on interpretation to a "match". We will refer here to a proposal of party *i* (denote by x_i) as the amount *i* demands for himself, and refer to x_1 and x_2 as matching iff $x_1 + x_2 \leq 1$. When an agreement is reached, party *i* receives x_i. (Note, that changing the interpretation of the match by requiring a perfect match, or changing a little the split of the pie by dividing the reminder, does not change the following results).

As in the alternating offers model, the players' preferences in the game are derived from player *i*'s utility function $U_i(x,t) = x_i \cdot \delta_i^t$ over the pairs (x,t).

The following are the possible histories:

$(x^1, x^2, ..., x^t)$ (for $t > 0$) after which both players have to make a new offer.

$(x^1, x^2, ..., x^t)$ (for $t > 0$) a terminal history evaluated by player *i* as $U_i(x,t) = x_i^t \cdot \delta_i^t$.

$(x^1, x^2, ..., x^t, ...)$ a terminal history evaluated by both players as 0.

First note, that in the one period model, every successful division of the whole pie (meaning that $x_1 + x_2 = 1$) is an equilibrium. Another (inefficient) equilibrium is the case where $x_1 = x_2 = 1$.

Using these one-shot possible equilibria, we can show that every division of the whole pie, in any one of the periods, can be sustained in SPE, as well as the case where there is no division of the pie at all. To see that a division (x_1^*, x_2^*) at period t^* can be sustained in equilibrium, consider the case where for every $t \neq t^*$ we have $x_1^t = x_2^t = 1$ and for $t = t^*$ we have $x_i^t = x_i^*$ for $i = 1,2$. Clearly, none of the parties can gain from deviation.

2. (Constant cost of bargaining)

The pair $x^*=(1,0)$ and $y^*=(1-c_1, c_1)$ is the unique SPE. It is straightforward to check that the strategy pair is SPE (by the one deviation property). In order to show uniqueness, let $M_i(G_i)$ be the supremum of a player i's payoff over the subgame perfect equilibria of subgames in which he makes the first proposal; let and $m_i(G_i)$ be the corresponding infimum for $i = 1, 2$.

We have $M_2(G_2) \le 1 - m_1(G_1) + c_1$, or (*) $m_1(G_1) \le 1 - M_2(G_2) + c_1$. Otherwise player 1 will reject the proposal and wait until the next period in which he can guarantee he gets a payoff of at least $m_1(G_1) - c_1$.

Now suppose that $M_2(G_2) \ge c_2$, under this assumption (**) $m_1(G_1) \ge 1 - M_2(G_2) + c_2$, since player 2 must accept the offer $M_2(G_2) - c_2$ because she cannot get a higher payoff in any continuation of the game.(*) together with (**) is a contradiction since $c_1 < c_2$. Thus $M_2(G_2) < c_2$. But this implies that $m_1(G_1) \ge 1$ since player 2 will accept any offer of player 1 (including getting zero) because by rejecting she will get a negative payoff. Also $m_1(G_1) \le 1$, so that $m_1(G_1) = 1$ and hence $M_1(G_1) = 1$.

As before (*), $M_2(G_2) \le 1 - m_1(G_1) + c_1$, so we have $M_2(G_2) \le c_1$; and $m_2(G_2) \ge c_1$ since player 1 will always accept payoff of 1- c_1 since by rejecting he will get less. So that $M_2(G_2) = m_2(G_2) = c_1$.

In every SPE of G_1 player 1's payoff is x_1^*. Player 2's payoff must be non negative and since $x_1^*+ x_2^*=1$, player 2's payoff is 0. This means that agreement is reached immediately. This means that in all SPE player 1 offers 1 and player 2 always offers c_1. Player 1 accepts offers which are at least good for him as c_1 and player 2 accepts offers which are at least as good for him as 0.

3.

We assume that u_1, u_2 are strictly monotonic and continuous. The proof is based on Osborne and Rubinstein page 123 with minor changes to fit the question.
Moreover x in the question which is a scalar is changed to be a vector of offers.

- Existence:

Consider the following as an equilibrium: Player 1 always offers the partition x^* and accepts any offer which gives him at least y^*. Player 2 always demands y^* and accepts any offer which gives him at least x^*. Since $\delta_2 \cdot u_2\left(y^*\right)=u_2\left(x^*\right)$ and $\delta_1 \cdot u_1\left(x^*\right)=u_1\left(y^*\right)$ both players cannot gain from a deviation.

- Uniqueness:

Let G_i be the game where *i* is the first one to give an offer.

let $M_i(G_i)$ be the supremum of a player i's payoff over the subgame perfect equilibria of subgames in which he makes the first proposal; let and $m_i(G_i)$ be the corresponding infimum for i = 1, 2.

Describe the pairs of payoffs on the Pareto frontier (which are efficient) by the function ϕ : if x is efficient then $u_2(x)=\phi(u_1(x))$.

Step 1. $m_2(G_2) \geq \phi(\delta_1 M_1(G_1))$.

In any SPE of G_2 Player 1 must accept an offer of $\delta_1 M_1$ since in the next stage her payoff will not be higher than M_1. Thus player 2's payoff cannot be lower than $\phi(\delta_1 M_1(G_1))$.

Step 2. $M_1(G_1) \leq \phi^{-1}(\delta_2 m_2(G_2))$

In any SPE of G_1 Player 2 must obtain a payoff of at least $\delta_2 m_2(G_2)$, since otherwise player 2 will reject the opening offer of player 1. Thus the payoff of player 1 cannot exceed $\phi^{-1}(\delta_2 m_2(G_2)$.

Step 3. $M_1(G_1)=u_1(x^*)$

We have that $M_1(G_1) \geq u_1(x^*)$ since there is a SPE of G_1 in which x* is agreed upon immediately. We will now show that $M_1(G_1) \leq u_1(x^*)$.

By the definition of ϕ we have that $\delta_2\phi(\delta_1 u_1((1,0)) > 0 = u_2(1,0) = \phi(u_1(1,0))$. Since ϕ is decreasing we get that $u_1(1,0) > \phi^{-1}(\delta_2\phi(\delta_1 u_1((1,0)))$. By step 1 and 2 we have $M_1(G_1) \leq \phi^{-1}(\delta_2\phi(\delta_1 M_1(G_1))$. Thus by the continuity of ϕ there exists $U_1 \in \left[M_1(G_1), u_1(1,0)\right]$ such that $U_1 = \phi^{-1}(\delta_2\phi(\delta_1 U_1))$ (a fixed point). If $M_1(G_1) > u_1(x^*)$ then $U_1 \neq u_1(x^*)$. Take a* and b* to be efficient agreements for which for $u_1(a^*) = U_1$ and $u_1(b^*) = \delta_1 u_1(a^*)$. By substituting $u_1(a^*)$ and $u_1(b^*)$ into $U_1 = \phi^{-1}\left(\delta_2\phi\left(\delta_1 U_1\right)\right)$ we get $u_1(a^*) = \phi^{-1}(\delta_2\phi(u_1(b^*)))$ which mean that $\phi(u_1(a^*)) = \delta_2\phi(u_1(b^*))$. But since a* and b* are efficient agreements we get $u_2(a^*) = \delta_2 u_2(b^*)$ contradicting the fact that x* and y* is the only pair that follows these conditions. The proofs for $m_1(G_1) = u_1(x^*)$, $M_2(G_2) = u_2(x^*)$ and $m_2(G_2) = u_2(x^*)$ is similar.

The remaining of the proof is the same as in the lecture notes.

4.

We will show that if $U_i(x,t) = u_i(x)\delta_i^t$ represents player *i*'s preference over the space $X \times T$ then so does $V_i(x,s) = v_i(x)\delta_i^t$, where $v_i(x) = u_i(x)^{\frac{\ln(\delta)}{\ln(\delta_i)}}$.

$$u_i(x)\delta_i^t \geq u_i(y)\delta_i^s \Leftrightarrow \left(u_i(x)\delta_i^t\right)^{\frac{\ln(\delta)}{\ln(\delta_i)}} \geq \left(u_i(y)\delta_i^s\right)^{\frac{\ln(\delta)}{\ln(\delta_i)}}$$

But,

$$\left(u_i(x)\delta_i^t\right)^{\frac{\ln(\delta)}{\ln(\delta_i)}} = u_i(x)^{\frac{\ln(\delta)}{\ln(\delta_i)}}(\delta_i^t)^{\frac{\ln(\delta)}{\ln(\delta_i)}} = u_i(x)^{\frac{\ln(\delta)}{\ln(\delta_i)}}(\delta_i^t)^{\log_{\delta_i}^{\delta}} =$$
$$= u_i(x)^{\frac{\ln(\delta)}{\ln(\delta_i)}}\delta_i^{\log_{\delta_i}^{\delta^t}} = u_i(x)^{\frac{\ln(\delta)}{\ln(\delta_i)}}\delta_i^{\log_{\delta_i}^{\delta^t}} = u_i(x)^{\frac{\ln(\delta)}{\ln(\delta_i)}}\delta_i^{\log_{\delta_i}^{\delta^t}} = u_i(x)^{\frac{\ln(\delta)}{\ln(\delta_i)}}\delta^t = v_i(x)\delta^t$$

So, $u_i(x)\delta_i^t \geq u_i(y)\delta_i^s \Leftrightarrow v_i(x)\delta^t \geq v_i(y)\delta^s$.

5.

The solution is based on Rubinstein, Safra and Thomson (1992), see also Binmore, Rubinstein and Wolinsky (1986).

For $x, y \in X$ we denote $x >> y$ if there is a player *i* and a number δ such that both $\delta \cdot u_i(x) > u_i(y)$ and $u_j(x) > \delta \cdot u_j(y)$. In other words, $x >> y$ if one of the players can "appeal" successfully against *y* by suggesting *x* for some discount factor δ.

Lemma (The single-peak property of >>):
Let $x, y, z \in X$ satisfy $z > x > y$. If not $[x >> z]$ then $x >> y$.

The full proof for the lemma appears in Rubinstein, Safra and Thomson (1992).

To get a slightly different intuition as to why the lemma is true, first note that $x >> y$ iff $u_1(x)\cdot u_2(x) > u_1(y)\cdot u_2(y)$. The first direction is obvious, to see that $u_1(x)\cdot u_2(x) > u_1(y)\cdot u_2(y) \Rightarrow \delta \cdot u_i(x) > u_i(y)$ *and* $u_j(x) > \delta \cdot u_j(y)$ note that if $u_1(x)\cdot u_2(x) > u_1(y)\cdot u_2(y)$ then for some i, $u_i(x) > u_i(y)$ and choose $\delta = (1+\varepsilon)\frac{u_i(y)}{u_i(x)}$ for a small enough ε. Therefore, saying that >> is "single peaked" is equivalent to saying that $f(x) = u_1(x)\cdot u_2(x)$ is a single peaked function. This claim is reasonable under the classic assumption regarding the functions u_i.

Next, note that there is no $x > x^*(\delta)$ (so that $x \succ_1 x^*(\delta)$) such that $x >> x^*(\delta)$ since if there was, the fact that not $[x^*(\delta) >> y^*(\delta)]$ together with the above lemma imply that $x^*(\delta) >> x$. Similarly, there is no $x < y^*(\delta)$ (so that $x \succ_2 y^*(\delta)$) such that $x >> y^*(\delta)$.

Therefore, for all δ, $x^*(\delta) \geq \arg\max_x u_1(x)\cdot u_2(x) \geq y^*(\delta)$. For any subsequence (δ_n) converging to 1 such that $x^*(\delta_n)$ and $y^*(\delta_n)$ converge to x^* and y^* respectively, it has to be true that $u_i(x^*) = u_i(y^*)$ for both *i* and thus $x^* = y^*$. Furthermore, $x^* \geq \arg\max_x u_1(x)\cdot u_2(x) \geq y^*$ and thus the sequence converge to $\arg\max_x u_1(x)\cdot u_2(x)$.

6. (Outside options)

It is straightforward to check that the strategy pair x*and y* is a subgame perfect equilibrium, where $x^* = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ and $y^* = \left(\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}, \frac{1-\delta_1}{1-\delta_1\delta_2}\right)$.

By assumption $y_1^* = \frac{\delta_1(1-\delta_1)}{1-\delta_1\delta_2} \succ d_1$ and $x_2^* = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2} \succ d_2$, since each player prefers the equilibrium payoff over the outside option.

Let M_1 and M_2 be the suprema of player 1's and player 2's payoffs over subgame perfect equilibria of the subgames in which players 1 and 2, respectively, make the first offer. Similarly, let m_1 and m_2 be the infima of these payoffs. We proceed in a number of steps in order to show uniqueness.

First observe that $m_1 \leq x_1^* \leq M_1$ and $m_2 \leq y_2^* \leq M_2$ (*)

Step 1. $m_2 \geq 1 - \max(\delta_1 M_1, d_1)$, since player 1 must always accept a payoff higher than his outside option and the maximal payoff he may receive next period .But $m_2 \geq 1 - \delta_1 M_1$ since $\delta_1 M_1 \geq \delta_1 x_1^* = y_1^* > d_1$

Step 2. $M_1 \leq 1 - \max\{d_2, \delta_2 m_2\}$

Since Player 2 obtains the payoff d_2 by opting out, we must have $M_1 \leq 1 - d_2$. The fact that $M_1 \leq 1 - \delta_2 m_2$ follows from the argument if player 1 offers player 2 a payoff of less than $\delta_2 m_2$ player 2 will always reject the offer and wait until the next period in which he can receive a payoff of at least that.

Step 3. $m_1 \geq 1 - \delta_2 M_2$, by the same argument in step 1.

Step 4. $M_2 \leq 1 - \max\{d_1, \delta_1 m_1\}$, by the same argument in step 2.

Step 5. $M_1 = m_1 = x_1^*$ and $M_2 = m_2 = y_2^*$.

By Step 2 we have $1 - M_1 \geq \delta_2 m_2$ (since both $1 - M_1 \geq \delta_2 m_2$ and $1 - M_1 \geq d_2$) and by Step 1 we have $m_2 \geq 1 - \delta_1 M_1$, that is $1 - M_1 \geq \delta_2 - \delta_1 \delta_2 M_1$, and hence

$M_1 \leq \dfrac{1-\delta_2}{1-\delta_1\delta_2} = x_1^*$. Hence $M_1 = x_1^*$ by (*).

Now, by Step 1 and substituting the result of M_1 we have $m_2 \geq 1 - \delta_1 M_1 = \dfrac{1-\delta_1}{1-\delta_1\delta_1}$.

Hence $m_2 = y_2^*$ by (*).

By Step 4 we have $1 - M_2 \geq \delta_1 m_1$ (since both $1 - M_2 \geq \delta_1 m_1$ and $1 - M_2 \geq d_1$) and by Step 3 we have $m_1 \geq 1 - \delta_2 M_2$, that is $1 - M_2 \geq \delta_1 - \delta_1\delta_2 M_2$, and hence

$M_2 \leq \frac{1-\delta_1}{1-\delta_1\delta_2} = y_2^*$. Hence $M_2 = y_2^*$ by (*).

Now, by Step 3 and substituting the result of M_2 we have $m_1 \geq 1 - \delta_2 M_2 = \frac{1-\delta_2}{1-\delta_1\delta_1}$.

Hence $m_1 = x_1^*$ by (*).

The remaining of the proof is the same as in the lecture notes.

7.

First note that if we wish to model the bargaining model with hyperbolic time preferences, we must take in account that the agent today is different then the game tomorrow. However, the game is the identical, and has the same equilibria. Furthermore, in the analysis we only use the relation between two consecutive periods (i.e. each type compares the payoffs with the payoffs of the following period, in case such period comes). Hence, we only care about the preference from a one period's difference point of view. This results in the next equilibrium:

$$x^* = \left(\frac{1-\beta\delta}{1-(\beta\delta)^2}, \frac{\beta\delta(1-\beta\delta)}{1-(\beta\delta)^2}\right),\ y^* = \left(\frac{\beta\delta(1-\beta\delta)}{1-(\beta\delta)^2}, \frac{1-\beta\delta}{1-(\beta\delta)^2}\right)$$

This equilibrium is identical to the one with the "regular" time preferences, where the δ is changed with $\beta\delta$, and is constant along all periods.

To see that this is an equilibrium, you can use the same arguments as in the class notes. To prove uniqueness, it is a little more complex….