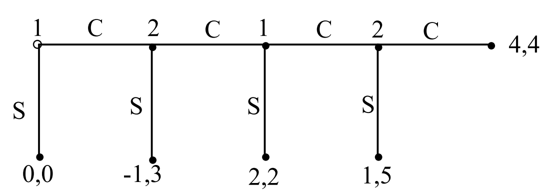
Solutions for problems set 6

1.

Each reduced strategy of player i corresponds to a set of (regular) strategies of player i in the extensive game. $(f_i) \rightarrow \{(s_i) | \text{ if } f_i(h) = s_i(h) \forall h \text{ in the domain of } f_i \}$. In this set, the outcomes for player i are the same, given the strategies of the other players since they result in the same terminal history. The set of Nash Equilibria of the strategic game with the reduced strategies corresponds to the set of (regular) Nash Equilibria of the extensive game in which the reduced strategies of each player corresponds to the set of (regular) strategies

Take for example the centipede game:



The set of strategies of both players is {(S,S),(S,C),(C,S),(C,C)}. The set of reduced strategies is {(S),(C,S),(C,C)}. The strategic form of the game is:

	SS	SC	CS	CC
SS	0,0*	0,0*	0,0	0,0
SC	0,0*	0,0*	0,0	0,0
CS	-1,3	-1,3	2,2	2,2
CC	-1,3	-1,3	1,5	4,4

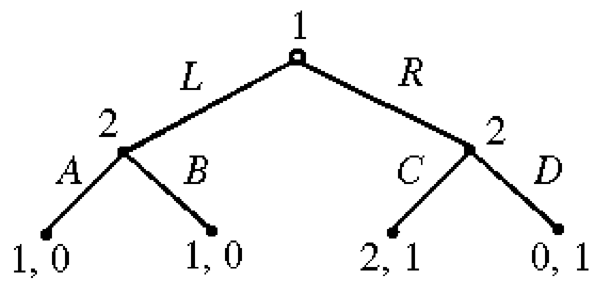
And the strategic form of the game with reduced strategies is:

	S	CS	CC
S	0,0*	0,0	0,0
CS	-1,3	2,2	2,2
CC	-1,3	1,5	4,4

The * stands for the Nash equilibria

2. (SPE of Stackelberg game)

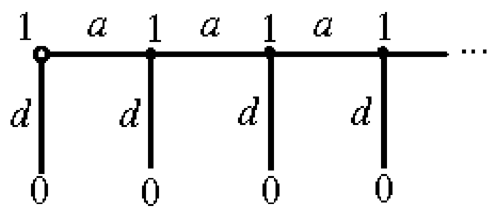
Consider the game



In this game (L, AD) is a subgame perfect equilibrium, with a payoff of (1, 0), while the solution of the maximization problem is (R, C), with a payoff of (2, 1).

3. (Necessity of finite horizon for one deviation property)

In the (one-player) game

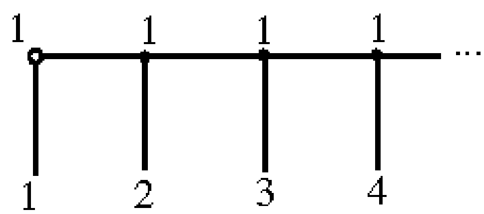


the strategy in which the player chooses d after every history satisfies the condition in Lemma of one deviation property but is not a subgame perfect equilibrium.

4. (Necessity of finiteness for Kuhn's theorem)

Consider the one-player game in which the player chooses a number in the interval [0, 1), and prefers larger numbers to smaller ones. That is, consider the game $\langle\{1\}, \{\phi\} \cup [0, 1), P, \{\succsim_1\}\rangle$ in which $P(\phi) = 1$ and $x \succ_1 y$ if and only if $x > y$. This game has a finite horizon (the length of the longest history is 1) but has no subgame perfect equilibrium (since [0, 1) has no maximal element).

In the infinite-horizon one-player game the beginning of which is shown in the following figure



the single player chooses between two actions after every history. After any history of length k the player can choose to stop and obtain a payoff of $k + 1$ or to continue; the payoff if she continues for ever is 0. The game has no subgame perfect equilibrium.

5. (SPE of games satisfying no indifference condition)

The hypothesis is true for all subgames of length one. Assume the hypothesis for all subgames with length at most k. Consider a subgame $\Gamma(h)$ with $l(\Gamma(h)) = k + 1$ and $P(h) = i$. For all actions a of player i such that $(h, a) \in H$ define $R(h, a)$ to be the outcome of some subgame perfect equilibrium of the subgame $\Gamma(h, a)$. By hypothesis all subgame perfect equilibria outcomes of $\Gamma(h, a)$ are preference equivalent; in a subgame perfect equilibrium of $\Gamma(h)$ player i takes an action that maximizes $\succsim_i$ over $\{R(h, a) : a \in A(h)\}$. Therefore player i is indifferent between any two subgame perfect equilibrium outcomes of $\Gamma(h)$; by the no indifference condition all players are indifferent among all subgame perfect equilibrium outcomes of $\Gamma(h)$.

We now show that the equilibria are interchangeable. For any subgame perfect equilibrium we can attach to every subgame the outcome according to the subgame perfect equilibrium if that subgame is reached. By the first part of the exercise the outcomes that we attach (or at least the rankings of these outcomes in the players' preferences) are independent of the subgame perfect equilibrium that we select. Thus by the one deviation property, any strategy profile s'' in which for each player i the strategy s_i'' is equal to either s_i or s_i' is a subgame perfect equilibrium.

6. (Armies)

We model the situation as an extensive game in which at each history at which player i occupies the island and player j has at least two battalions left, player j has two choices: conquer the island or terminate the game. The first player to move is player 1. (We do not specify the game formally.)

We show that in every subgame in which army i is left with y_i battalions ($i = 1, 2$) and army j occupies the island, army i attacks if and only if either $y_i > y_j$, or $y_i = y_j$ and y_i is even.

The proof is by induction on $\min\{y_1, y_2\}$. The claim is clearly correct if

$\min\{y_1, y_2\} \leq 1$. Now assume that we have proved the claim whenever $\min\{y_1, y_2\} \leq m$ for some $m \geq 1$. Suppose that $\min\{y_1, y_2\} = m+1$. There are two cases.

- either $y_i > y_j$, or $y_i = y_j$ and y_i is even: If army i attacks then it occupies the island and is left with y_i–1 battalions. By the induction hypothesis army j does not launch a counterattack in any subgame perfect equilibrium, so that the attack is worthwhile.
- either $y_i < y_j$, or $y_i = y_j$ and y_i is odd: If army i attacks then it occupies the island and is left with y_i–1 battalions; army j is left with y_j battalions. Since either $y_i - 1 < y_j - 1$ or $y_i - 1 = y_j - 1$ and is even, it follows from the inductive hypothesis that in all subgame perfect equilibria there is a counterattack. Thus army i is better off not attacking.

Thus the claim is correct whenever $\min\{y_1, y_2\} \leq m + 1$, completing the inductive argument.

7.

First lets recall the conclusion from the lecture notes that in the finite version of chess there is a SPE and hence also a NE. Because chess is strictly competitive, the equilibrium payoff is unique and thus any NE strategy of a player guarantees the player his equilibrium payoff. Thus either White has a strategy that guarantees that he wins, or Black has a strategy that guarantees that she wins, or each player has a strategy that guarantees that the outcome of the game is a draw.
Assume the payoff of the outcome win is 1, of loose is -1 and draw is 0. According to the official law of the chess game it is no longer a finite game. However we will show

that in any SPE the players will either choose to stop the game or will be indifferent between a draw and continuing the game to infinity.

If in the finite version the White payer can guarantee that she wins (meaning that no position was repeated 3 times) than this is true also for the official version of chess and the value of the game is 1. The same is true if the Black player can guarantee he wins and the value of the game is -1.

If however in the finite version the SPE follows a history in which a position is repeated 3 times the value of the game is 0 (Note that in the subgame that starts in that position for the first time, all other action for the player whose turn it is to move must lead to a draw or a loss).

Assume that a position is repeated 3 times and it is white's time to move. Any action that leads to a finite subgame will end with a draw or a loss (as in the finite game). Also if she chooses "draw" then the game ends and we are back to the finite version where the game has a value. However if she chooses to continue to a subgame which is not finite it must be the case that she weakly prefers this over a draw. Since this is a zero sum game is must be the case that the Black player weakly prefers an outcome of a draw over continuing the game forever. But during the history that repeats this position over and over there must be another position which is repeated at least three times in which the Black player can call a draw. So it is either the case that both player continue for ever and are indifferent between this and a draw or that one of the player ends the game with a draw. In any case the value of this subgame is 0.

8. (ODP and Kuhn's theorem with chance moves)

One deviation property: The argument is the same as in the proof of the one deviations property.

Kuhn's theorem: The argument is the same as in the proof of the existence of a subgame perfect equilibrium with the following addition. If $P(h^*) = c$ then $R(h^*)$ is the lottery in which $R(h^*, a)$ occurs with probability $f_c(a \mid h)$ for each $a \in A(h^*)$.

9. (Three players sharing pie)

The game is given by

- $N = \{1, 2, 3\}$
- $H = \{\phi\} \cup X \cup \{ (x, y): x \in X$ and $y \in \{yes, no\} \times \{yes, no\}$ where $X = \{x \in \mathbb{R}^3_+ : \sum_{i=1}^{3} x_i = 1\}$
- $P(\phi) = 1$ and $P(x) = \{2, 3\}$ if $x \in X$
- for each $i \in N$ we have $(x, (yes, yes)) \succ_i (z, (yes, yes))$ if and only if $x_i > z_i$; if $(A, B) \neq (yes, yes)$ then $(x, (yes, yes)) \succ_i (z, (A, B))$ if $x_i > 0$ and $(x, (yes, yes)) \sim_i (z, (A, B))$ if $x_i = 0$; if $(A, B) \neq (yes, yes)$ and $(C, D) \neq (yes, yes)$ then $(x, (C, D)) \sim_i (z, (A, B))$ for all $x \in X$ and $z \in X$.

In each subgame that follows a proposal x of player 1 there are two types of Nash equilibria. In one equilibrium, which we refer to as $Y(x)$, players 2 and 3 both accept x. In all the remaining equilibria the proposal x is not implemented; we refer to the set of these equilibria as $N(x)$. If both $x_2 > 0$ and $x_3 > 0$ then $N(x)$ consists of the single equilibrium in which players 2 and 3 both reject x. If $x_i = 0$ for either $i = 2$ or $i = 3$, or both, then $N(x)$ contains in addition equilibria in which a player who is offered 0 rejects the proposal and the other player accepts the proposal.

Consequently the equilibria of the entire game are the following.

- For any division x, player 1 proposes x. In the subgame that follows the proposal x of player 1, the equilibrium is $Y(x)$. In the subgame that follows any proposal y of player 1 in which $y_1 > x_1$, the equilibrium is in $N(y)$. In the subgame that follows any proposal y of player 1 in which $y_1 < x_1$, the equilibrium is either $Y(y)$ or is in $N(y)$.
- For any division x, player 1 proposes x. In the subgame that follows any proposal y of player 1 in which $y_1 > 0$, the equilibrium is in $N(y)$. In the subgame that follows any proposal y of player 1 in which $y_1 = 0$, the equilibrium is either $Y(y)$ or is in $N(y)$.

10. (Naming numbers)

The game is given by

- $N = \{1, 2\}$
- $H = \{\phi\} \cup \{Stop, Continue\} \cup \{(Continue, y) : y \in Z \times Z\}$ where Z is the set of nonnegative integers
- $P(\phi) = 1$ and $P(Continue) = \{1, 2\}$
- the preference relation of each player is determined by the payoffs given in the question.

In the subgame that follows the history *Continue* there is a unique subgame perfect equilibrium, in which both players choose 0. Thus the game has a unique subgame perfect equilibrium, in which player 1 chooses *Stop* and, if she chooses *Continue*, both players choose 0.

Note that if the set of actions of each player after player 1 chooses *Continue* were bounded by some number M then there would be an additional subgame perfect equilibrium in which player~1 chooses *Continue* and each player names M, with the payoff profile (M^2,M^2).