Solution for Problem Set 4

1. (Guess the average)

Let k^* be the largest number to which any player's strategy assigns positive probability in a mixed strategy equilibrium and assume that player i's strategy does so. We now argue as follows.

- In order for player i's strategy to be optimal his payoff from the pure strategy k^* must be equal to his equilibrium payoff.
- In any equilibrium player i's expected payoff is positive, since for any strategies of the other players he has a pure strategy that for some realization of the other players' strategies is at least as close to 2/3 of the average number as any other player's number.
- In any realization of the strategies in which player i chooses k^* and receives a positive payoff, some other player also chooses k^*, since by the previous two points player i's payoff is positive in this case, so that no other player's number is closer to 2/3 of the average number than k^*. (Note that all the other numbers must be less than 2/3 of the average number.)
- In any realization of the strategies in which player i chooses $k^* \geq 1$, he can increase his payoff by choosing $k^* - 1$, since by making this change he becomes the outright winner rather than tying with at least one other player.

The remaining possibility is that $k^* = 1$: every player uses the pure strategy in which he announces the number 1.

2. (Guessing right)

In the game each player has K actions; $u_1(k, k) = 1$ for each $k \in \{1, ..., K\}$ and $u_1(k, l) = 0$ if $k \neq l$. The strategy pair $((1/K, ..., 1/K), (1/K, ..., 1/K))$ is the unique mixed strategy equilibrium, with an expected payoff to player 1 of $1/K$. To see this, let (p^*, q^*) be a mixed strategy equilibrium. If $p^*_k > 0$ then the optimality of the action k for player 1 implies that q^*_k is maximal among all the q_l^*, so that in particular $q^*_k > 0$, which implies that p^*_k is minimal among all the p^*_l, so that $p_k^* \leq 1/K$. Hence $p^*_k = 1/K$ for all k; similarly $q_k = 1/K$ for all k.

3. (Air strike)

The payoffs of player 1 are given by the matrix

$$\begin{pmatrix} 0 & v_1 & v_1 \\ v_2 & 0 & v_2 \\ v_3 & v_3 & 0 \end{pmatrix}$$

Let (p^*, q^*) be a mixed strategy equilibrium.

Step 1. If $p_i^* = 0$ then $q_i^* = 0$ (otherwise q^* is not a best response to p^*); but if $q_i^* = 0$ and $i \leq 2$ then $p_{i+1} = 0$ (since player i can achieve v_i by choosing i). Thus if for $i \leq 2$ target i is not attacked then target i+1 is not attacked either.

Step 2. $p^* \neq (1,0,0)$: it is not the case that only target 1 is attacked.

Step 3. The remaining possibilities are that only targets 1 and 2 are attacked or all three targets are attacked.

- If only targets 1 and 2 are attacked the requirement that the players be indifferent between the strategies that they use with positive probability implies that $p^* = (v_2/(v_1 + v_2), v_1/(v_1 + v_2), 0)$ and $q^* = (v_1/(v_1 + v_2), v_2/(v_1 + v_2), 0)$. Thus the expected payoff of Army A is $v_1v_2/(v_1 + v_2)$. Hence this is an equilibrium if $v_3 \leq v_1v_2/(v_1 + v_2)$.
- If all three targets are attacked the indifference conditions imply that the probabilities of attack are in the proportions $v_2v_3 : v_1v_3 : v_1v_2$ and the probabilities of defense are in the proportions $z - 2v_2v_3 : z - 2v_3v_1 : z - 2v_1v_2$ where $z = v_1v_2 + v_2v_3 + v_3v_1$. For an equilibrium we need these three proportions to be nonnegative, which is equivalent to $z - 2v_1v_2 \geq 0$, or $v_3 \geq v_1v_2/(v_1 + v_2)$.

4. (Investment race)

The set of actions of each player i is $A_i = [0,1]$. The payoff function of player i is

$$u_i(a_1,a_2) = \begin{cases} -a_i & \text{if } a_i < a_j \\ 1/2 - a_i & \text{if } a_i = a_j \end{cases}$$

where $j \in \{1,2\} \setminus \{i\}$.

We can represent a mixed strategy of a player i in this game by a probability distribution function F_i on the interval [0, 1], with the interpretation that $F_i(v)$ is the probability that player i chooses an action in the interval $[0, v]$. Define the *support* of F_i to be the set of points v for which $F_i(v + \varepsilon) - F_i(v - \varepsilon) > 0$ for all $\varepsilon > 0$, and define v to be an *atom* of F_i if $F_i(v) > \lim_{\varepsilon \to 0} F_i(v - \varepsilon)$. Suppose that (F_1^*, F_2^*) is a mixed strategy Nash equilibrium of the game and let S_i^* be the support of F_i^* for $i = 1, 2$.

Step 1. $S^*_1 = S^*_2$.

Proof. If not then there is an open interval, say (v, w), to which F_i^* assigns positive probability while F_j^* assigns zero probability (for some i, j). But then i can increase his payoff by transferring probability to smaller values within the interval (since this does not affect the probability that he wins or loses, but increases his payoff in both cases).

Step 2. If v is an atom of F_i^* then it is not an atom of F_j^* and for some $\varepsilon > 0$ the set S_j^* contains no point in $(v - \varepsilon, v)$.

Proof. If v is an atom of F_i^* then for some $\varepsilon > 0$, no action in $(v - \varepsilon, v]$ is optimal for player j since by moving any probability mass in F_i^* that is in this interval to either $v + \delta$ for some small $\delta > 0$ (if $v < 1$) or 0 (if $v = 1$), player j increases his payoff.

Step 3. If $v > 0$ then v is not an atom of F_i^* for $i = 1, 2$.

Proof. If $v > 0$ is an atom of F_i^* then, using Step 2, player i can increase his payoff by transferring the probability attached to the atom to a smaller point in the interval $(v - \varepsilon, v)$.

Step 4. $S_i^* = [0, M]$ for some $M > 0$ for $i = 1, 2$.

Proof. Suppose that $v \notin S_i^*$ and let $w^* = \inf\{w|\ w \in S_i^*$ and $w > v\}$. By Step 1 we have $w^* \in S_j^*$, and hence, given that w^* is not an atom of F_i^* by Step 3, we require j's payoff at w^* to be no less than his payoff at v. Hence $w^* = v$. By Step 2 at most one distribution has an atom at 0, so $M > 0$.

Step 5. $S^*_i = [0,1]$ and $F_i^*(v) = v$ for $v \in [0,1]$ and $i = 1, 2$.

Proof. By Steps 2 and 3 each equilibrium distribution is atomless, except possibly at 0, where at most one distribution, say F_i^*, has an atom. The payoff of i at $v > 0$ is $F_j^*(v) - v$, where $i \neq j$. Thus the constancy of i's payoff on $[0,M]$ and $F^*_j(0) = 0$ requires that $F_j^*(v) = v$, which implies that $M = 1$. The constancy of j's payoff then implies that $F_i^*(v) = v$.

We conclude that the game has a unique mixed strategy equilibrium, in which each player's probability distribution is uniform on [0,1].

5.

For a realistic (and somewhat horrifying) motivation for this question see the New York Times article "The Case of Kitty Genovese" at http://www.garysturt.free-online.co.uk/The%20case%20of%20Kitty%20Genovese.htm .

In the symmetric mixed strategy Nash equilibrium every individual calls the police with probability p, and each is indifferent between calling the police or not.
If an individual calls the police her payoff is $1-c$ (we assume that $0 < c < 1$) with probability 1. If she doesn't call her expected payoff is $1 \cdot \left(1-(1-p)^{N-1}\right)$. From the indifference condition we have $1-c = 1-(1-p)^{N-1} \Rightarrow p = 1-c^{\frac{1}{N-1}}$.
This implies that the probability that at least one of the individuals calls the police is $P(N) = 1-(1-p)^N = 1-\left(1-1+c^{\frac{1}{N-1}}\right)^N = 1-c^{\frac{N}{N-1}}$. Hence, $P(1) = 1$ and $P(N)$ is

monotonic decreasing in *N*, with asymptotic value of $\lim_{N\to\infty} P(N)=1-c$, implying that it is socially worse to have more individuals at the scene.

6.

	L	R
L	0,0	1,2
R	2,1	0,0

- To analyze the set of mixed strategy NE denote the probability of a player to choose L by *p*. We require that each player be indifferent between playing L and R so $p\cdot 0+(1-p)\cdot 1=p\cdot 2+(1-p)\cdot 0 \Rightarrow p=\frac{1}{3}$. To see that this is also an ESS first note that when interacting with a "normal" player, the expected payoff of every type is $\frac{2}{3}$ regardless of the strategy he uses. Now, assume that a mutant with a strategy of $\left(\frac{1}{3}+\delta,\frac{2}{3}-\delta\right)$ with $\left(-\frac{1}{3}<\delta<\frac{2}{3}\right)$ enters the economy in proportion $\varepsilon>0$. In this case the expected payoff of the mutant is
$$(1-\varepsilon)\cdot\frac{2}{3}+\varepsilon\cdot\left[\left(\frac{1}{3}+\delta\right)\cdot\left(\frac{2}{3}-\delta\right)\cdot 1+\left(\frac{2}{3}-\delta\right)\cdot\left(\frac{1}{3}+\delta\right)\cdot 2\right]=$$
$$=(1-\varepsilon)\cdot\frac{2}{3}+\varepsilon\cdot\left[\frac{6}{9}+\delta-3\delta^2\right]$$
which is strictly lower than the expected payoff of a "normal" type
$$(1-\varepsilon)\cdot\frac{2}{3}+\varepsilon\cdot\left[\frac{1}{3}\cdot\left(\frac{2}{3}-\delta\right)\cdot 1+\frac{2}{3}\cdot\left(\frac{1}{3}+\delta\right)\cdot 2\right]=$$
$$=(1-\varepsilon)\cdot\frac{2}{3}+\varepsilon\cdot\left[\frac{6}{9}+\delta\right]$$
- Denote a player *i* pure strategy by the vector $\left\langle x_1^i,\left(x_L^i,x_R^i\right)\right\rangle$ where $x_1^i\in\{L,R\}$ is player *i*'s action in the first period and $x_Y^i\in\{L,R\}$ is player *i*'s action given that in the first period player $j\neq i$ chose $Y\in\{L,R\}$. An 8x8 matrix can describe all possible results of the two-period game. However, if we denote a mixed strategy in the two-period game by $\left\langle \alpha_1^i,\left(\alpha_{LL}^i,\alpha_{LR}^i,\alpha_{RL}^i,\alpha_{RR}^i\right)\right\rangle$ (α_{XY}^i is the

probability that player i chooses L given that she chose $X \in \{L, R\}$ and player $j \neq i$ chose $Y \in \{L, R\}$ in the first period) it is straight forward to see that the two possible symmetric equilibria are: $\left\langle \frac{1}{3}, \left(\frac{1}{3}, 0, 1, \frac{1}{3}\right)\right\rangle$ and $\left\langle \frac{1}{3}, \left(\frac{1}{3}, 1, 0, \frac{1}{3}\right)\right\rangle$.

7.

a. The pure strategy equilibria are (D,L,A), (T,R,A), (D,L,C), and (T,R,C).

b. A correlated equilibrium with the outcome described is given by: $\Omega = \{x, y\}$, $\pi(x) = \pi(y) = 1/2$; $P_1 = P_2 = \{\{x\}, \{y\}\}$, $P_3 = \Omega$; $\sigma_1(\{x\}) = T$, $\sigma_1(\{y\}) = B$; $\sigma_2(\{x\}) = L$, $\sigma_2(\{y\}) = R$; $\sigma_3(\Omega) = B$. Note that player 3 knows that (T,L) and (D,R) will occur with equal probabilities, so that if she deviates to A or C she obtains $3/2 < 2$.

c. If player 3 were to have the same information as players 1 and 2 then the outcome would be one of those predicted by the notion of Nash equilibrium, in all of which she obtains a payoff of zero.

8.

Each player can choose to climb the mountain or not. Denote the strategy profile by (p_1, p_2, p_3), where p_i is the probability that player i climbs the mountain.

The mixed strategy Nash equilibria of the game depend on c:

- c=0: (1,1,p) where $p \in [0,1]$
- $0 < c \leq \frac{1}{2}$ (two possible equilibria):

    1. (p,1,1-p) or (1,p,1-p) where p=1-2c.

    2. (p,p,1-p) where $p = \sqrt{1-2c}$ (see demonstration below).

- $c = \frac{1}{2}$: (p,0,q) or (0,p,q) where $p, q \in [0,1]$
- $c > \frac{1}{2}$: (0,0,0)

It is easy to verify that these are indeed Nash equilibria. We will demonstrate this for the case: (x,y,z) where $x, y, z \in (0,1)$. Since all actions are chosen with a positive probability all players must be indifferent between climbing the mountain or not. This implies that :

Player 1:

$$\underbrace{y + (1-y)z/2 + (1-y)(1-z) - c}_{\text{player 1's payoff if he climbs the mountain}} = \underbrace{yz/2 + y(1-z) + (1-y)(1-z)/2}_{\text{player 1's payoff if he don't climb the mountain}}$$

Player 2:

$$\underbrace{x + (1-x)z/2 + (1-x)(1-z) - c}_{\text{player 2's payoff if he climbs the mountain}} = \underbrace{xz/2 + x(1-z) + (1-x)(1-z)/2}_{\text{player 2's payoff if he don't climb the mountain}}$$

From these two equations we get that if z<1 then x=y and $c = 1/2 - x(1-z)/2$

Player 3:

$$2x(1-x)/2 + (1-x)^2 - c = (1-x)^2/2 \Rightarrow c = 1/2 - x^2/2$$

This indicates that if all players choose both actions with a positive probability then the Nash equilibrium must be (x,x,1-x) where $x = \sqrt{1-2c}$ and c<1/2.

In this case the probability that player 3 wins is:

$$2(1-x)^2 x/2 + (1-x)^2 x/2 + (1-x)^3 > 0.5$$

This implies that player 3 might have a better (larger than 0.5) probability of winning when $x^3 - 3x + 1 > 0$. This is true for $x < \frac{1}{3}$ which occurs when $4/9 \le c \le 0.5$ (approximately).