Solution for Problem set 3

1.

Let $p(\omega_k)$ denote the common prior of both players and v_k is the value of the object in state ω_k. The expected value that player *i* assigns to the object when he receives the signal $t_i \in T_i$ is:

(*) $E_i(v|t_i) = \sum_{\omega_k \in \tau_i^{-1}(t_i)} \frac{p(\omega_k)}{p(\tau_i^{-1}(t_i))} v_k$, where $E_i(v|t_i) > v^* \; \forall t_i \in T_i$.

The expected value of the object for player *i* (before receiving the signal) is:

$$E_i(v) = E_{t_i}(E_i(v|t_i)) = \sum_{t_i \in T_i} p(\tau_i^{-1}(t_i)) \sum_{\omega_k \in \tau_i^{-1}(t_i)} \frac{p(\omega_k)}{p(\tau_i^{-1}(t_i))} v_k = \sum_{\omega_k \in \Omega} p(\omega_k) v_k .$$

According to (*), $E_i(v)$ must be larger than v^* since it is a weighted average of numbers that are all larger than v^*.

Similarly, the expected value player *j* assigns to the object when he receives the signal $t_j \in T_j$ is: $E_j(v|t_j) < v^*, \forall t_j \in T_j$ and the expected value of the object for player *j* is: $E_j(v) = \sum_{\omega_k \in \Omega} p(\omega_k) v_k$ where $E_j(v) < v^*$. Observe however that

(**) $E_i(v) = \sum_{\omega_k \in \Omega} p(\omega_k) v_k = E_j(v)$, which is a contradiction.

Note that if we relax the assumption of the common prior (**) need not hold in equality.

2.

Let $s^* = (s^*(t_1),....,s^*(t_k))$ be NE of G_2 and assume that $(a_{(1,t_i)} = s^*(t_i))_{i=1,..,k}$ is not a NE of G_1. Then there exists a signal $t_i \in T$ and an action $b \in A$ such that:

$$\sum_{\omega \in \tau^{-1}(t_i)} \frac{p(\omega)}{p(\tau^{-1}(t_i))} u(s^*(t_i),\omega) < \sum_{\omega \in \tau^{-1}(t_i)} \frac{p(\omega)}{p(\tau^{-1}(t_i))} u(b,\omega) \Rightarrow$$
$$\sum_{\omega \in \tau^{-1}(t_i)} p(\omega) u(s^*(t_i),\omega) < \sum_{\omega \in \tau^{-1}(t_i)} p(\omega) u(b,\omega) .$$

But if this is the case then

$$v(s^*) = \sum_{\omega\in\Omega} p(\omega)u(s^*(\tau(\omega)),\omega) = \sum_{\omega\in\tau^{-1}(t_k),\forall t_k \neq t_i} p(\omega)u(s^*(\tau^{-1}(\omega)) + \sum_{\omega\in\tau^{-1}(t_i)} p(\omega)u(s^*(t_i),\omega) <$$
$$\sum_{\omega\in\tau^{-1}(t_k),\forall t_k \neq t_i} p(\omega)u(s^*(\tau^{-1}(\omega)),\omega) + \sum_{\omega\in\tau^{-1}(t_i)} p(\omega)u(b,\omega) = v(s^*(t_1),\ldots,b,\ldots s^*(t_k))$$

which is a contradiction to s^* being a NE in the G_2. The proof in the other direction is similar.

3. (More information may hurt)

Consider the Bayesian game in which $N = \{1, 2\}$, $\Omega = \{\omega_1, \omega_2\}$, the set of actions of player 1 is $\{U, D\}$, the set of actions of player 2 is $\{L, M, R\}$, player 1's signal function is defined by $\tau_1(\omega_1) = 1$ and $\tau_1(\omega_2) = 2$, player 2's signal function is defined by $\tau_2(\omega_1) = \tau_2(\omega_2) = 0$, the belief of each player is (1/2, 1/2), and the preferences of each player are represented by the expected value of the payoff function defined as follows (where $0 < \varepsilon < 1/2$).

State ω_1:

	L	*M*	*R*
U	1, 2ε	1, 0	1, 3ε
D	2, 2	0, 0	0, 3

State ω_2:

	L	*M*	*R*
U	1, 2ε	1, 3ε	1, 0
D	2, 2	0, 3	0, 0

This game has a unique Nash equilibrium ((*D*,*D*), *L*) (that is, both types of player 1 choose *D* and player 2 chooses *L*). The expected payoffs at the equilibrium are (2, 2).

In the game in which player 2, as well as player 1, is informed of the state, the unique Nash equilibrium when the state is ω_1 is (*U*, *R*); the unique Nash equilibrium when the state is ω_2 is (U, M). In both cases the payoff is $(1, 3\varepsilon)$, so that player 2 is worse off than he is when he is ill-informed.

4. (Exchange game)

In the Bayesian game there are two players, say $N = \{1, 2\}$, the set of states is $\Omega = S \times S$, the set of actions of each player is {*Exchange*, *Don't exchange*}, the signal function of each player i is defined by $\tau_i(s_1, s_2) = s_i$, and each player's belief on Ω is that generated by two independent copies of F . Each player's preferences are represented by the payoff function $u_i((X, Y), \omega) = \omega_j$ if $X = Y =$ *Exchange* and $u_i((X, Y), \omega) = \omega_i$ otherwise.

Let x be the smallest possible prize and let M_i be the highest type of player i that chooses *Exchange*. If $M_i > x$ then it is optimal for type x of player j to choose *Exchange*. Thus if $M_i \geq M_j$ and $M_i > x$ then it is optimal for type M_i of player i to choose *Don't exchange*, since the expected value of the prizes of the types of player j that choose *Exchange* is less than M_i. Thus in any possible Nash equilibrium $M_i = M_j = x$: the only prizes that may be exchanged are the smallest.

5.

- In the Bayesian game there are n players, say $N = \{1,2,...,n\}$, the set of states is $\Omega = V \times V \times ... \times V$, the set of actions of each player is the set of possible bids $B \equiv [0,\infty)$, the signal function of each player i is defined by $\tau_i(v_1, v_2,..., v_n) = v_i$, and each player's belief on Ω is that generated by independent copies of F . Each player's preferences are represented by the expectation of the payoff function $u_i((b_1,...,b_n),\omega) = \omega_i - \max_{j\neq i} b_j$ if b_i is one of the highest bids and i is the lowest index in this set of highest bids, and $u_i((b_1,...,b_n),\omega) = 0$ otherwise.
- Assume that for every i, $b_i(v_i) = v_i$. Player j expected payoff conditional on his signal v_j is

  $$w_j((v_1,..,b_j,..,v_n),\omega) = \Pr(player\ \ j\ wins\ the\ auction \mid b_j)\cdot E\left(v_j - \max_{i\neq j} b_i \mid player\ \ j\ wins\ the\ auction\right)$$

  Clearly, if $b_j(v_j) = v_j$ player j cannot profit by deviating: Given any possible state in which he wins the object than he has a non-negative payoff, his payoff may differ only if he lowers his bid so that he doesn't win, but in this case he didn't raise his payoff. Similarly, given any possible state in which he doesn’t win the object his payoff is zero and might change only if he raises his offer and win, but then he will have to pay at least his valuation of the object and have a non-positive payoff. Thus, in all cases the expected payoff for player *j* under $b_j(v_j)=v_j$ is at least as high as under any other strategy.

However, this is not the unique BNE. Consider for example that the bidding strategies are as follows: for one player *i*, $b_i(v_i)=\max_{v\in V} v$ and for every $j \neq i$, $b_j(v_j)=0$. It is quite straightforward to see that this is also an equilibrium.

- The same argument as in the previous subsection works without assuming that for every *i*, $b_i(v_i)=v_i$. Hence, for each type to bid his valuation is a weakly dominant action.