Lecture L-2: Zero Sum Games

Readings: Osborne and Rubinstein Ch 2.5

Strictly Competitive Games

Let us discuss now a class of games in which there are two players, whose preferences are diametrically opposed. For convenience assume $N = \{1,2\}$.

A strategic game $\langle \{1,2\}, (A_i), (\succsim_i) \rangle$ is *strictly competitive* if for any $a \in A$ and $b \in A$ we have $a \succsim_1 b$ if and only if $b \succsim_2 a$.

A strictly competitive game is sometimes called *zero-sum* because if player 1's preference relation $\succsim_1$ is represented by the payoff function u_1 then player 2's preference relation is represented by $u_2 = -u_1$.

We identify a pattern of strategic reasoning of a special kind. We say that player *i* *maxminimizes* if he chooses an action that is best for him under the assumption that whatever he does, player *j* will choose his action to hurt him as much as possible.

We interpret it in two possible ways. (1) A decision making method: the player always assume the worst and try to minimize the disaster. (2) A strategic reasoning: in spite of the simultaneousness, a player anticipates that his opponent will respond optimally (from the opponent's point of view).

Main message: We will show that a strictly competitive game possesses a Nash equilibrium, a pair of actions is a Nash equilibrium if and only if the action of each player is a maxminimizer.

This provides a link between individual decision-making and the reasoning behind the notion of Nash equilibrium. It will follow that for strictly competitive games that possess Nash equilibria all equilibria yield the same payoffs.

Definition: Let $\langle\{1,2\},(A_i),(\succsim_i)\rangle$ be a strictly competitive strategic game. Let $\succsim_i$ be represented by a payoff function u_i. Without loss of generality, assume that $u_2 = -u_1$.

The action $z^* \in A_1$ is a maxminimizer for player 1 if $\min_{y\in A_2} u_1(z^*,y) \geq \min_{y\in A_2} u_1(x,y)$ $\forall x \in A_1$. That is, a maxminimizer for player i is an action that maximizes the payoff that player i can *guarantee*.

Lemma The maxminimization of player 2's payoff is equivalent to the minmaximization of player 1's payoff. That is, let $\langle\{1,2\},(A_i),(u_i)\rangle$ be a strictly competitive strategic game.

(a) $\max_{y\in A_2} \min_{x\in A_1} u_2(x,y) = -\min_{y\in A_2} \max_{x\in A_1} u_1(x,y)$.

(b) $y \in A_2$ solves the problem $\max_{y\in A_2} \min_{x\in A_1} u_2(x,y)$ iff it solves the problem $\min_{y\in A_2} \max_{x\in A_1} u_1(x,y)$.

Proof Note that for any function f we have $\min_z(-f(z)) = -\max_z f(z)$ and $\arg\min_z(-f(z)) = \arg\max_z f(z)$.

Thus, for every $y \in A_2$ $-\min_{x\in A_1} u_2(x,y) = \max_{x\in A_1}(-u_2(x,y)) = \max_{x\in A_1} u_1(x,y)$.

$\max_{y\in A_2} \min_{x\in A_1} u_2(x,y) = -\min_{y\in A_2}[-\min_{x\in A_1} u_2(x,y)] = -\min_{y\in A_2} \max_{x\in A_1} u_1(x,y)$;

in addition $y \in A_2$ is a solution of the problem $\max_{y\in A_2} \min_{x\in A_1} u_2(x,y)$ if and only if it is a solution of the problem $\min_{y\in A_2} \max_{x\in A_1} u_1(x,y)$.

Proposition Let $G = \langle\{1,2\},(A_i),(u_i)\rangle$ be a strictly competitive strategic game.

(a) If (x^*,y^*) is a Nash equilibrium of G then x^* is a maxminimizer for player 1 and y^* is a maxminimizer for player 2.

(b) If (x^*,y^*) is a Nash equilibrium of G then $\max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y) = u_1(x^*,y^*)$, and thus all Nash equilibria of G yield the same payoffs.

(c) If $\max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y)$ (and thus, in particular, if G has a Nash equilibrium (see part b)), x^* is a maxminimizer for player 1, and y^* is a maxminimizer for player 2, then (x^*, y^*) is a Nash equilibrium of G. proposition

Proof (a) and (b).

Let (x^*, y^*) be a Nash equilibrium of G.

Then $u_2(x^*, y) \leq u_2(x^*, y^*)$ for all $y \in A_2$ or, since $u_2 = -u_1$, $u_1(x^*, y^*) \leq u_1(x^*, y)$ for all $y \in A_2$.

Hence $\min_y u_1(x^*, y) = u_1(x^*, y^*)$

For any $x \in A_1$ we have $\min_y u_1(x, y) \leq u_1(x, y^*)$.

Since (x^*, y^*) be a Nash equilibrium of G we have $u_1(x, y^*) \leq u_1(x^*, y^*)$ for all $x \in A_1$. Thus $u_1(x^*, y^*) = \max_x \min_y u_1(x, y)$ and x^* is a maxminimizer for player 1.

An analogous argument for player 2 establishes that y^* is a maxminimizer for player 2 and $u_2(x^*, y^*) = \max_y \min_x u_2(x, y)$.

By the Lemma $u_1(x^*, y^*) = -u_2(x^*, y^*) = -\max_y \min_x u_2(x, y) = \min_y \max_x u_1(x, y)$.

Proof of (c):

Let $v^* = \max_x \min_y u_1(x, y) = \min_y \max_x u_1(x, y)$.

By the Lemma we have $\max_y \min_x u_2(x, y) = -v^*$.

Since x^* is a maxminimizer for player 1 we have $u_1(x^*, y) \geq v^*$ for all $y \in A_2$;

Since y^* is a maxminimizer for player 2 we have $u_2(x, y^*) \geq -v^*$ and thus $u_1(x, y^*) \leq v^*$ for all $x \in A_1$.

Letting $y = y^*$ and $x = x^*$ in these two inequalities we obtain $u_1(x^*, y^*) = v^*$

Using the fact that $u_2(x^*, y^*) = -u_1(x^*, y^*)$, we conclude that (x^*, y^*) is a Nash equilibrium of G.

▸ By (c) a Nash equilibrium can be found by solving the problem $\max_x \min_y u_1(x, y)$.

▸ By (a) and (c) Nash equilibria of a strictly competitive game are *interchangeable*: if (x, y) and (x', y') are equilibria then so are (x, y') and (x', y).

▸ Always $\max_x \min_y u_1(x, y) \leq \min_y \max_x u_1(x, y)$

since $u_1(x', y) \leq \max_x u_1(x, y)$ for all y,

and thus $\min_y u_1(x', y) \leq \min_y \max_x u_1(x, y)$ for all x.

▸ In *Matching Pennies*, $\max_x \min_y u_1(x, y) = -1 < \min_y \max_x u_1(x, y) = 1$.

▸(b) shows that $\max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y)$ for any 0-sum game that has NE.

If $\max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y)$ then we say that this payoff, the equilibrium payoff of player 1, is the *value of the game*.

Problem set G-2

1. (**Exercise**) Let G be a strictly competitive game that has a Nash equilibrium.

▲ Show that if some of player 1's payoffs in G are increased in such a way that the resulting game G' is strictly competitive then G' has no equilibrium in which player 1 is worse off than she was in an equilibrium of G. (Note that G' may have no equilibrium at all.)

▲ Show that the game that results if player 1 is prohibited from using one of her actions in G does not have an equilibrium in which player 1's payoff is higher than it is in an equilibrium of G.

▲ Give examples to show that neither of the above properties necessarily holds for a game that is not strictly competitive.

2. (**Exercise**)

▲ What can you say anout the Nash equilibrium of a symmetric zero-sum game?

▲ Invent a formal concept which will state that in a zero-sum game where each player has to choose an action from a set X (the same action set to both players), player 1 is in a better position.

3. (**Exercise**) Cosnider the following game. Player 1 has to state a number of 20 digits and player 2 has to repeat on the number. If he succeed player 2 wins the game, if he fails player 1 wins the game.

Analyse the situation as a zero sum game. What is the value of the game. Would you prefer to be player 1 or 2 in this game? Comment on what is missing from the model.