Problem Set G-1: Nash Equilibrium

1. First price auction

The set of actions of each player i is $[0, \infty)$ (the set of possible bids) and the payoff of player i is $v_i - b_i$ if his bid b_i is equal to the highest bid and no player with a lower index submits the same bid, and 0 otherwise.

The set of Nash equilibria is the set of profiles b of bids with $b_1 \in [v_2, v_1]$, $b_j \leq b_1$ for all $j \neq 1$, and $b_j = b_1$ for some $j \neq 1$.

It is easy to verify that all these profiles are Nash equilibria. To see that there are no other equilibria, first we argue that there is no equilibrium in which player 1 does not obtain the object. Suppose that player $i \neq 1$ submits the highest bid b_i and $b_1 < b_i$. If $b_i > v_2$ then player i's payoff is negative, so that he can increase his payoff by bidding 0. If $b_i \leq v_2$ then player 1 can deviate to the bid b_i and win, increasing his payoff.

Now let the winning bid be b^*. We have $b^* \geq v_2$, otherwise player 2 can change his bid to some value in (v_2, b^*) and increase his payoff. Also $b^* \leq v_1$, otherwise player 1 can reduce her bid and increase her payoff. Finally, $b_j = b^*$ for some $j \neq 1$ otherwise player 1 can increase her payoff by decreasing her bid.

2. Second price auction

The set of actions of each player i is $[0, \infty)$ (the set of possible bids) and the payoff of player i is $v_i - b_j$ if his bid b_i is equal to the highest bid and b_j is the highest of the other players' bids (possibly equal to b_i) and no player with a lower index submits this bid, and 0 otherwise.

For any player i the bid $b_i = v_i$ is a dominant action. To see this, let x_i be another action of player i. If $\max_{j\neq i} b_j \geq v_i$ then by bidding x_i player i either does not obtain the object or receives a nonpositive payoff, while by bidding b_i he guarantees himself a payoff of 0. If $\max_{j\neq i} b_j < v_i$ then by bidding v_i player i obtains the good at the price $\max_{j\neq i} b_j$, while by bidding x_i either he wins and pays the same price or loses.

An example of equilibrium in which player $j \neq 1$ obtains the good is that in which $b_1 < v_j$, $b_j > v_1$, and $b_i = 0$ for all players $i \notin \{1,j\}$.

3. War of attrition

The set of actions of each player i is $A_i = [0,\infty)$ and his payoff function is

$$u_i(t_1,t_2) = \begin{cases} -t_i & \text{if } t_i < t_j \\ v_i/2 - t_i & \text{if } t_i = t_j \end{cases}$$

where $j \neq i$ Let (t_1, t_2) be a pair of actions. If $t_1 = t_2$ then by conceding slightly later than t_1 player 1 can obtain the object in its entirety instead of getting just half of it, so this is not an equilibrium. If $0 < t_1 < t_2$ then player 1 can increase her payoff to zero by deviating to $t_1 = 0$. Finally, if $0 = t_1 < t_2$ then player 1 can increase her payoff by deviating to a time slightly after t_2 unless $v_1 - t_2 \leq 0$. Similarly for $0 = t_2 < t_1$ to constitute an equilibrium we need $v_2 - t_1 \leq 0$. Hence (t_1, t_2) is a Nash equilibrium if and only if either $0 = t_1 < t_2$ and $t_2 \geq v_1$ or $0 = t_2 < t_1$ and $t_1 \geq v_2$.

Comment An interesting feature of this result is that the equilibrium outcome is independent of the players' valuations of the object.

4. Location game

There are n players, each of whose set of actions is $\{Out\} \cup [0,1]$. (Note that the model differs from Hotelling's in that players choose whether or not to become candidates.) Each player prefers an action profile in which he obtains more votes than any other player to one in which he ties for the largest number of votes; he prefers an outcome in which he ties for first place (regardless of the number of candidates with whom he ties) to one in which he stays out of the competition; and he prefers to stay out than to enter and lose.

Let F be the distribution function of the citizens' favorite positions and let

$m = F^{-1}(1/2)$ be its median (which is unique, since the density f is everywhere positive).

It is easy to check that for $n = 2$ the game has a unique Nash equilibrium, in which both players choose m.

The argument that for $n = 3$ the game has no Nash equilibrium is as follows.

- There is no equilibrium in which some player becomes a candidate and loses, since that player could instead stay out of the competition. Thus in any equilibrium all candidates who enter must tie for first place.
- There is no equilibrium in which a single player becomes a candidate, since by choosing the same position any of the remaining players ties for first place.
- There is no equilibrium in which two players become candidates, since by the argument for $n = 2$ in any such equilibrium they must both choose the median position m, in which case the third player can enter close to that position and win outright.
- There is no equilibrium in which all three players become candidates:
    - If all three choose the same position then any one of them can choose a position slightly different and win outright rather than tying for first place;

- If two choose the same position while the other chooses a different position then the lone candidate can move closer to the other two and win outright.
- If all three choose different positions then (given that they tie for first place) either one of the extreme candidates can move closer to his neighbor and win outright.

Following the same type of arguments as in the case where n=3, we have that when n=4, the only equilibrium possible is if all players enter and they all tie. Similarly, as before it is not possible that all 4 are in the same location. In addition, in the rightmost location there can not be a single candidate, since then this candidate can move left and win (a similar argument holds for the leftmost candidate). Consequently, the only equilibrium possible is two players located in the one place and the other two in another, and all candidates tie. In this case, the right couple of candidates should be located not to the left of $m = F^{-1}(3/4)$, otherwise one of the rightmost candidates can move to the right, get more than ¼ of the votes and win, they also can not be located to the right of $m = F^{-1}(3/4)$, or one of them can move to the left and win. By this argument, and a similar argument for the left couple of candidates we get that in equilibrium one couple is located at $m = F^{-1}(3/4)$, and the other at $m = F^{-1}(1/4)$. Note that this can only be an equilibrium if $F^{-1}(1/2)=(F^{-1}(1/4)+ F^{-1}(3/4))/2$, otherwise there is no equilibrium for n=4 as well.

Comment If the density f is not everywhere positive then the set of medians may be an interval, say [m^*, m^{**}]. In this case the game has Nash equilibria when $n = 3$; in all equilibria exactly two players become candidates, one choosing m^* and the other choosing m^{**}.

5. A demand game

The set of actions of each player i is [0, 1] (the set of possible demands) and the payoff of player i is:

- In game a: $x_i + \frac{(1 - x_i - x_{-i})}{2}$ if $x_i + x_{-i} \leq 1$, and 0 otherwise.

- In game b: $C + x_i + \frac{(1 - x_i - x_{-i})}{2}$ if $x_i + x_{-i} \leq 1$ and $x_i \leq x_{-i}$ (C>1),

  $x_i + \frac{(1 - x_i - x_{-i})}{2}$ if $x_i + x_{-i} \leq 1$ and $x_i > x_{-i}$, and C otherwise.

- In game c: $x_i + \frac{(1 - x_i - x_{-i})}{2} + \lambda\left[x_{-i} + \frac{(1 - x_i - x_{-i})}{2}\right]$ if $x_i + x_{-i} \leq 1$, and 0 otherwise.

In game a, x_1 and x_2 are equilibrium demands if $x_1 + x_2 = 1$ or if both players demand 1.

In game b, x_1 and x_2 are equilibrium demands if $x_1 = x_2 \leq 1/2$ or if both players demand 1.

In game c ($\lambda \neq 0$):

- $1 > \lambda > 0$: x_1 and x_2 are equilibrium demands iff $x_1 + x_2 = 1$.
- $\lambda = 1$: Every x_1 and x_2 for which $x_1 + x_2 \leq 1$ are equilibrium demands.
- If $\lambda > 1$: the only equilibrium demands are $(0,0)$.

Otherwise, if $\lambda<0$, x_i and x_{-i} are equilibrium if $x_1 + x_2 = 1$ and for every *i*, $x_i + \frac{(1 - x_i - x_{-i})}{2} + \lambda\left[x_{-i} + \frac{(1 - x_i - x_{-i})}{2}\right] \geq 0$, or if $x_1 + x_2 > 1$ and for every *i*, $1 - x_{-1} + \lambda x_{-i} \leq 0$.

6. Ducks

The set of players N={1,…,12}, the set of possible actions is A={1,2} and the payoffs of a player are $\frac{1}{\#\{\text{players that choose } 1\}}$ if he chooses 1, and $\frac{2}{\#\{\text{players that choose } 2\}}$ if he chooses 2.

It is straightforward to see that all equilibria will be of the same type, 8 players will choose 2 and 4 will choose 1.

7. Symmetric games

Define the function $F : A_1 \to A_1$ by $F(a_1) = B_2(a_1)$ (the best response of player 2 to a_1). The function F satisfies the conditions of Kakutani's fixed point theorem, and hence has a fixed point, say a_1*. The pair of actions (a_1*, a_1*) is a Nash equilibrium of the game since, given the symmetry, if a_1* is a best response of player 2 to a_1* then it is also a best response of player 1 to a_1*.

A symmetric finite game that has no symmetric equilibrium is *Hawk--Dove*.