Solution of the Exam in Micro A 02/2006

Question 1

(a) We show that the following preference relation induces the behavior specified in the question:

$$u(x) = \begin{cases} 1 + x_{K+1} & if\ x > D \\ \min\left(\frac{x_1}{d_1}, \frac{x_2}{d_2}, ..., \frac{x_K}{d_K}\right) & otherwise \end{cases}$$

where $D = (d_1, d_2 ..., d_K)$.

Since $\min\left(\frac{x_1}{d_1}, \frac{x_2}{d_2}, ..., \frac{x_K}{d_K}\right) \leq 1$ for all bundles in the relevant domain, whenever the consumer can afford more than D he will choose to purchase the bundle D and spend the rest of his income on x_{K+1}. If he cannot afford more than D then the consumer will behaves according the min function and consume the bundle tD $(t \leq 1)$ (where t is a function of his budget).

(b) We will show that behavior does not induce a continuos preference relation (and in particular does not induce a continuous monotonic and convex preference relation).

According to the consumer's behavior $(d_1, d_2, ..., d_K, \varepsilon) \succ (d_1, d_2, ..., d_K, 0)$ however for every $\delta > 0$ $(d_1 - \delta, d_2, ..., d_K, \varepsilon) \prec (d_1, d_2, ..., d_K, 0)$ contradicting continuity.

Question 2

Let $x \sim_i y$ denote that individual i views x as "the same as" y and $x \nsim_i y$ denote that individual i views x as "not the same as" y, $x \sim y$ and $x \nsim y$ with no index is society's opinion. Let $(\sim_i)_{i=1,...,N}$ be a profile of equivalence relations then an aggregation method is a function $F : (\sim_i)_{i=1,...,N} \to E$.

(a)

- P: For all $x, y \in X$ and every profile $(\sim_i)_{i=1,...,N}$, if $x \sim_i y$ $\forall i$ then $x \sim y$ and if $x \nsim_i y$ $\forall i$ then $x \nsim y$.
- I*: For every $a, b, c, d \in X$ and any two profiles $(\sim_i)_{i=1,...,N}$ and $(\sim'_i)_{i=1,...,N}$ if for all i, $a \sim_i b$ iff $c \sim'_i d$ then $a \sim b$ iff $c \sim' d$.

(b)

- Satisfies P but not I*: $F((\sim_i)_{i=1,...,N})$ is the most common equivalence relation among $(\sim_i)_{i=1,...,N}$ (with some tie breaker).

P is satisfied since if all individuals view x as "the same as" y then in particular the most common equivalence relations view x as "the same as" y, thus society views x as "the same as" y (the opposite is true if x is "not the same as" y for all i).

I* is not satisfied: look at the following example: $X = \{x, y, z\}$ Let $(\sim_i)_{i=1,\ldots,5}$ be $x \sim_i y \sim_i z$ for $i = 1, 2$, $x \nsim_3 y$ and $x \sim_3 z$, $x \nsim_4 y$ and $y \sim_4 z$ and $x \nsim_5 y \nsim_5 z \nsim_5 x$. The the most common equivalence relation is that of $i = 1, 2$ so $x \sim_i y$. However for $(\sim'_i)_{i=1,\ldots,5}$ where $\sim'_i = \sim_i$for $i = 1, 2$ and $\sim'_i = \sim_5$ for $i = 3, 4, 5$ we have $x \nsim y$, even though $x \sim_i y$ iff $x \sim'_i y$ for all i, contradicting I*.

- Satisfies I* but not P: $F((\sim_i)_{i=1,\ldots,N})$ is $x \sim y \sim z$ for every profile $(\sim_i)_{i=1,\ldots,N}$.

  I* is satisfied since $F((\sim_i)_{i=1,\ldots,N})$ is constant for every profile.

  P is not satisfied since even if $x \nsim_i y \; \forall i$, $x \sim y$ contradicting P.

- Satisfies I* and P: $\forall x, y \in X \; x \sim y$ iff $x \sim_i y \; \forall i \in G \subset N$.

  P is satisfied since if all individuals view x as "the same as" y then in particular $\forall i \in G$ view x as "the same as" y and thus society views x as "the same as" y. If all individuals view x as "not the same as" y then $\exists i \in G$ that views x as "not the same as" y thus society views x as "not the same as" y.

  I* is satisfied since in any two profiles $(\sim_i)_{i=1,\ldots,N}$ and $(\sim'_i)_{i=1,\ldots,N}$ if for all i, $a \sim_i b$ iff $c \sim'_i d$ we have $a \sim_i b$ iff $c \sim'_i d \; \forall i \in G$ thus $a \sim b$ iff $c \sim' d$.

  (c) We will show that if X includes at least three elements, then the only aggregation method which satisfies P and I* is the aggregation method which determines a subset $G^* \subseteq N$ such that $x \sim y$ iff $x \sim_i y \; \forall i \in D$.

Define $\Gamma = \{G \subseteq N \;|$for all $x, y \in X,$ if for all $i \in G \; x \sim_i y$ and for all $j \notin G \; x \nsim_j y$ then $x \sim y\}$. Note that if $G \in \Gamma$ then $G \neq \varnothing$ since $G = \varnothing$ would imply that $x \sim y$ for the profile $(\sim_i)_{i=1,\ldots,N}$ in which $x \nsim_i y \; \forall i \in N$ contradicting P. Furthermore, Γ is not empty since by P $N \in \Gamma$.

If G_1,$G_2 \in \Gamma$ then $G_1 \cap G_2 \in \Gamma$.

We have to show that for any x, y and for any profile $(\sim'_i)_{i=1,\ldots,N}$ for which $x \sim'_i y$ for all $i \in G_1 \cap G_2$, and $x \nsim'_i y$ for all $i \in G_1 \cap G_2$ the equivalence relation $F((\sim'_i)_{i=1,\ldots,N})$ determines that $x \sim' y$. By I^* it is sufficient to show that for one pair a and b, and for one profile $(\sim_i)_{i=1,\ldots,N}$ that agrees with the profile $(\sim'_i)_{i=1,\ldots,N}$ on the pair $\{a, b\}$, the equivalence relation $F((\sim_i)_{i=1,\ldots,N})$ determines that $a \sim b$. Let $c \neq a, b$. Let $(\sim_i)_{i=1,\ldots,N}$ be a profile satisfying for all $i \in G_1 \cap G_2$, $a \sim_i b \sim_i c$, for all $i \in G_1 \backslash G_2 \; a \nsim_i b$ and $a \sim_i c$, for all $i \in G_2 \backslash G_1$ $a \nsim_i b$ and $b \sim_i c$ and for all $i \in N \backslash (G_1 \cup G_2)$ $a \nsim_i b$ and $b \nsim_i c$. Since $G_1 \in \Gamma$ $a \sim c$. Since $G_2 \in \Gamma$ $b \sim c$. By transitivity $a \sim b$.

There exists a unique minimal (with respect to inclusion) non empty subset $G^* \in \Gamma$.

Assume there does not exist a unique minimal subset $G^* \in \Gamma$. Then there exist two subsets G_1,G_2 $\in \Gamma$ such that $G_1 \neq G_2$ and $\nexists G \in \Gamma$ such that $G \subset G_1$ and $\nexists G \in \Gamma$ such that $G \subset G_2$. By the previous claim $G_1 \cap G_2 \in \Gamma$ and $G_1 \cap G_2 \subset G_1$ and $G_1 \cap G_2 \subset G_1$, a contradiction. Furthermore, $G^* \neq \varnothing$ since $N \in \Gamma$ and $\varnothing \notin \Gamma$.

If $G \in \Gamma$ then for all $G' \supseteq G$ $G' \in \Gamma$. Take any $x, y \in X$. Let $(\sim_i)_{i=1,\ldots,N}$ be a profile satisfying for all $i \in G$ $x \sim_i y \sim_i z$, for all $i \in G' \backslash G$ $x \sim_i y$ and $x \nsim_i z$ and for all $i \in N \backslash G'$ $x \nsim_i y$, $y \nsim_i z$. Since $G \in \Gamma$ $x \sim z$ and $y \sim z$. By transitivity a $x \sim y$ thus $G' \in \Gamma$. This implies that for any profile $(\sim'_i)_{i=1,\ldots,N}$ for which for all $i \in G^*$ $x \sim'_i y$ then $x \sim' y$.

We are left to show that if $x \sim' y$ then for all $i \in G^*$, $x \sim'_i y$. Assume not. Then there exists $i \in G^*$ such that $x \nsim'_i y$. By P we have that $\exists j \in N - \{i\}$ such that $x \sim'_j y$. Let $G' = \{j \,|x \sim'_i y\}$ so $i \notin G'$ and $G' \in \Gamma$. But if $G' \in \Gamma$ then $G^* \subseteq G'$ contradicting $i \in D$.

Question 3

Note that in any equilibrium since the payoff function is linear t_i satisfies for all i:

$$(*) \quad t_i = \begin{cases} 0 & if\ v_i < V \\ [0,1] & if\ v_i = V \\ 1 & v_i > V \end{cases}$$

Since v_i is monotonically increasing, it follows that if $t_i > 0$ then then $t_j = 1$ for all $j > i$ and $t_j = 0$ for all $j < i$.

(a)

- For the case where $N = 10$ and $v_i = i$ the following is an equilibrium: $t_i = 0$ for $i < 5$, $t_i = 1$ for $i \geq 5$ and $V = \sum_{i=5}^{10} \frac{v_i}{10} = 4.5$. This is an equilibrium. According to $(*)$ t_i optimal for all i and $V = \sum_{i=1}^{N} \frac{t_i v_i}{N}$.
- For the case where $N = 3$ and $v_1 = 1$, $v_2 = 2$ and $v_3 = 5$ the following is an equilibrium: $t_1 = 0$, $t_2 = 0.5$, $t_3 = 1$ $V = \frac{0.5v_2+v_3}{3} = 2$. Again, according to $(*)$ t_i optimal for all i and $V = \sum_{i=1}^{N} \frac{t_i v_i}{N}$.

For both cases the equilibrium is unique, since for every $V' > V$, $\sum_{i=1}^{N} \frac{t_i^{'} v_i}{N} \leq \sum_{i=1}^{N} \frac{t_i v_i}{N}$ implying that $V' \neq \sum_{i=1}^{N} \frac{t_i v_i}{N}$ (likewise for $V' < V$).

(b) Proof of existence of an equilibrium in the general case:

Let $N = 1$, then $t_1 = 1$ and $V = v_1$ is an equilibrium.

Let $N > 1$. Define $g(i) = \sum_{j=i}^{N} \frac{v_j}{N}$ and $v(i) = v_i$ for $i = 1, ..., N$. ($g(i)$ is the average product per person when only $j \geq i$ work). Notice that $g(i)$ is strictly decreasing and $v(i)$ is strictly increasing in i, furthermore $g(1) > v(1)$ and $g(N) > v(N)$. So there must exist an i^* such that $g(i^* + 1) \leq v(i^* + 1)$ and $g(i^*) > v(i^*)$. There are two possible cases: $g(i^*) > g(i^* + 1) \geq v(i^*)$ and $g(i^*) > v(i^*) > g(i^* + 1)$.

- For $g(i^*) > g(i^* + 1) \geq v(i^*)$: let $V = \sum_{j=i^*+1}^{N} \frac{v_j}{N} = g(i^* + 1)$, $t_j = 1$ for all $j \geq i^* + 1$ and $t_j = 0$ for all $j < i^* + 1$. This is an equilibrium since for all $j \geq i^* + 1$ $V = g(i^* + 1) \leq v(i^* + 1) \leq v(j)$ thus $t_j = 1$ is optimal and for all $j < i^* + 1$ $g(i^* + 1) \geq v(i^*) \geq v(j)$ thus $t_j = 0$ is optimal. Finally $V = \sum_{j=i^*+1}^{N} \frac{v_j}{N} = \sum_{i=1}^{N} \frac{t_i v_i}{N}$.
- For $g(i^*) > v(i^*) > g(i^* + 1)$: In this case i^* separates the market into those who do not work ($j < i^*$) and the rest who work by choosing t^* such that $\sum_{j=i^*+1}^{N} \frac{v_j}{N} + \frac{t^* v_{i^*}}{N} = v_{i^*}$. Let $V = \sum_{j=i^*+1}^{N} \frac{v_j}{N} + \frac{t^* v_{i^*}}{N} = v_{i^*}$, $t_j = 1$ for all $j \geq i^* + 1$, $t_{i^*} = t^*$ and $t_j = 0$ for all $j < i^*$.This is an equilibrium since for all $j \geq i^* + 1$, $V = v_{i^*} < v_j$ thus $t_j = 1$ is optimal, for $j = i^*$ $V = v_{i^*}$ thus $t_{i^*} = t^*$ and for all $j < i^*$ $V = v_{i^*} > v_j$ thus $t_j = 0$ is optimal. Finally, $V = \sum_{j=i^*+1}^{N} \frac{v_j}{N} + \frac{t^* v_{i^*}}{N} = \sum_{i=1}^{N} \frac{t_i v_i}{N}$.