NYU 2005 exam solution

Problem 1. (inspired by Chen, M.K., V.Lakshminarayanan and L.Santos (2005))

In an experiment a monkey is given $m = 12$ *coins. The monkey faces* m *consecutive choices. In each instance he gives one coin to either one of two experimenters, one who is holding* a *apples and one who is holding* b *bananas.*

(1) Assume that the experiment is repeated with different values of a *and* b *and that every time the monkey trades the first* 4 *coins for apples and then trades the next* 8 *coins for bananas. The experimenter claims that the monkey's choices confirm consumer theory.*

Show that the above monkey's behavior is indeed consistent with the classical assumptions of consumer behavior (namely, that his behavior can be explained as the maximization of a monotonic, continuous, convex preference relation on the space of bundles).

We can model the problem of the monkey as the following. It has wealth m and can buy two commodities, apples and bananas The price of one apple is $p_a = \frac{1}{a}$ and the price of one banana is $p_b = \frac{1}{b}$. So what the experiment shows is that the monkey always expends 1/3 of its wealth in apples and 2/3 in bananas. But this is exactly the behavior we would observe from a consumer that had Cobb-Douglas preferences with weights 1/3 and 2/3.

(2) Assume that later it was observed that when the monkey holds an arbitrary number m *of coins, then independent of* a *and* b*, he exchanges first* 4 *coins for apples and then exchanges the remaining* $m - 4$ *coins for bananas. Is this behavior consistent with the consumer model?*

The behavior of the monkey now is not consistent with consumer theory. That is, there is no preference relation that rationalizes this behavior. To see this assume that $\succsim$ is monotonic and $\succsim$ rationalizes the monkey behavior. Let $m = 4$, $a = 2$ and $b = 4$. In this case the monkey would buy the following bundle $(A = 8, B = 0)$. Since the monkey could also buy $(A = 4, B = 8)$, it must be the case that $(A = 8, B = 0) \succsim (A = 4, B = 8)$ Monotonicity of $\succsim$ implies that $(A = 9, B = 1) \succ (A = 8, B = 0) \succsim (A = 4, B = 8) \succsim (A = 4, B = 6)$. Now let's look at the monkey's choice when $m = 10$, $a = 1$ and $b = 1$. In this case the monkey would chose $(A = 4, B = 6)$, but it contradicts the fact that the monkey's choice is rationalized by $\succsim$, since it has enough resources to buy $(A = 9, B = 1)$.

Problem 2.

A consumer lives in a world of K *commodities. He holds classical preferences over those commodities. The goods are split into two categories, 1 and 2, of* K_1*and* K_2 *goods, respectively* ($K_1 + K_2 = K$)*. The consumer receives two types of money:* w_1 *units of wealth which can be exchanged for goods in the first* *category only and w_2 units of wealth which can be exchanged for goods in the second category only.*

Define the induced preference relation over the two-dimensional space (w_1, w_2). Show that those preferences are monotonic, continuous and convex.

We can write the consumer's problem as the following

$$x^*(w_1, w_2) = \max \succsim (x_1, x_2)$$
$$s.t\ p_1 x_1 \leq w_1 \text{ and } p_2 x_2 \leq w_2$$

We can define our preference relation over the space (w_1, w_2) as

$$(w_1, w_2) \succsim^* (w_1', w_2') \Leftrightarrow x^*(w_1, w_2) \succsim x^*(w_1', w_2')$$

Now let's prove the properties of $\succsim^*$.

Monotonicity: Let $(w_1', w_2') \gg (w_1, w_2)$. This implies that

$$p_1 x_1 (w_1, w_2) < w_1' \text{ and } p_2 x_2 (w_1, w_2) < w_2'$$

So there exists $(y_1, y_2) \gg x(w_1, w_2)$ such that

$$p_1 y_1 < w_1' \text{ and } p_2 y_2 < w_2'$$

Monotonicity of $\succsim$ implies that $(y_1, y_2) \succ x(w_1, w_2)$. Finally, notice that, since (y_1, y_2) satisfies the budget constraint of the problem (w_1', w_2'), it must be the case that $x(w_1', w_2') \succsim (y_1, y_2) \succ x(w_1, w_2)$, which implies that $(w_1', w_2') \succ^* (w_1, w_2)$.

Convexity: Let $(w_1', w_2') \succsim^* (w_1, w_2)$. This implies that $x(w_1', w_2') \succsim x(w_1, w_2)$. Let $w^\lambda = \lambda (w_1', w_2') + (1-\lambda)(w_1, w_2)$ and $x^\lambda = \lambda x(w_1', w_2') + (1-\lambda) x(w_1, w_2)$ for some $\lambda \in [0,1]$. We know that

$$\begin{aligned} p_1 x_1 (w_1, w_2) &\leq w_1,\ p_2 x_2 (w_1, w_2) \leq w_2 \\ p_1 x_1 (w_1', w_2') &\leq w_1',\ p_2 x_2 (w_1', w_2') \leq w_2' \end{aligned}$$

which implies that

$$p_1 x_1^\lambda \leq w_1^\lambda,\ p_2 x_2^\lambda \leq w_2^\lambda$$

Moreover,convexity of $\succsim$ implies that $x^\lambda \succsim x(w_1, w_2)$. Therefore, it must be the case that $x\left(w_1^\lambda, w_2^\lambda\right) \succsim x^\lambda \succsim x(w_1, w_2)$, which implies that $\left(w_1^\lambda, w_2^\lambda\right) \succsim^* (w_1, w_2)$.

Continuity: To show this we'll first show that the demand function x is continuous. So, let $(w_1^n, w_2^n) \to (w_1, w_2)$, we need to show that $x(w_1^n, w_2^n) \to x(w_1, w_2)$. Let's first show that the sequence $x(w_1^n, w_2^n)$ is inside of some compact set. Define

$$\overline{w_1} = \sup \{w_1^n\},\ \overline{w_2} = \sup \{w_2^n\}$$

It's clear that for any n we must have

$$x\left(w_1^n, w_2^n\right) \in \left\{x \in \mathbb{R}_+^{k_1+k_2} : p_1 x_1 \leq \overline{w_1} \text{ and } p_2 x_2 \leq \overline{w_2}\right\}$$

So our sequence is indeed entirely contained in a compact set. Now suppose that $x\left(w_1^n, w_2^n\right) \nrightarrow x\left(w_1, w_2\right)$. This implies that there exists subsequence $x\left(w_1^k, w_2^k\right)$, such that $x\left(w_1^k, w_2^k\right) \rightarrow y \neq x\left(w_1, w_2\right)$. The fact that $x\left(w_1^k, w_2^k\right) \rightarrow y$ implies that

$$p_1 y_1 \leq w_1 \text{ and } p_2 y_2 \leq w_2$$

So it must be the case that $x\left(w_1, w_2\right) \succ \left(y_1, y_2\right)$. Continuity of $\succsim$ implies that there exists $z \ll x\left(w_1, w_2\right)$ such that $z \succ \left(y_1, y_2\right)$. But then, continuity of $\succsim$ together with the fact that $p_1 z_1 < w_1$ and $p_2 z_2 < w_2$, imply that for k large enough

$$p_1 z_1 < w_1^k,\ p_2 z_2 < w_2^k \text{ and } z \succ x\left(w_1^k, w_2^k\right)$$

which contradicts the optimality of $x\left(w_1^k, w_2^k\right)$. Now that we know that the demand function is continuous we can easily show that $\succsim^*$ is continuous. Let $\left(w_1^n, w_2^n\right) \rightarrow \left(w_1, w_2\right)$ and $\left(\widehat{w}_1^n, \widehat{w}_2^n\right) \rightarrow \left(\widehat{w}_1, \widehat{w}_2\right)$ be such that $\left(w_1^n, w_2^n\right) \succsim^* \left(\widehat{w}_1^n, \widehat{w}_2^n\right)\ \forall n$. This implies that $x\left(w_1^n, w_2^n\right) \succsim x\left(\widehat{w}_1^n, \widehat{w}_2^n\right)\ \forall n$. By continuity of $\succsim$ and continuity of the demand function x,.this implies that $x\left(w_1, w_2\right) \succsim x\left(\widehat{w}_1, \widehat{w}_2\right)$, which is equivalent to $\left(w_1, w_2\right) \succsim^* \left(\widehat{w}_1, \widehat{w}_2\right)$.

Problem 3

Let X be a finite set containing at least three elements. In this question a choice function C assigns for every non-empty subset A a set $C(A) \subseteq A$ which is non-empty.

Consider the following "Choice Axiom ":

If $A, B \subseteq X$, $B \subseteq A$ and $C(A) \cap B \neq \emptyset$, then $C(B) = C(A) \cap B$.

a) Show that the Choice Axiom is equivalent to the existence of a preference relation $\succsim$ such that $C(A) = \{x \in A | x \succsim a \text{ for all } a \in A\}$.

It's enough to show that the Choice Axiom in the question is equivalent to the Weak Axiom.

Choice Axiom$\Rightarrow$WA. Let $x \in C(A)$, $y \in C(B)$ and $x, y \in A \cap B$. Define $B' = A \cap B$. Clearly $C(A) \cap B' \neq \emptyset$ and $C(B) \cap B' \neq \emptyset$, so the Choice Axiom implies that $C(A) \cap B = C(A) \cap B' = C(B') = C(B) \cap B' = C(B) \cap A$. But x clearly is in $C(A) \cap B$, which implies that $x \in C(B) \cap A$, which itself implies that $x \in C(B)$.

WA$\Rightarrow$Choice Axiom. Let $B \subseteq A$ and $C(A) \cap B \neq \emptyset$. Let $x \in C(B)$ and $y \in C(A) \cap B$. Clearly $x, y \in A \cap B$. But then the WA implies that $x \in C(A)$ and, therefore, $x \in C(A) \cap B$ and $y \in C(B)$. But this means that $C(B) \subseteq C(A) \cap B$ and $C(A) \cap B \subseteq C(B)$, which is equivalent to $C(B) = C(A) \cap B$.

b)Consider a weaker axiom:

If $A, B \subseteq X$, $B \subseteq A$ and $C(A) \cap B \neq \emptyset$, then $C(B) \subseteq C(A) \cap B$.

Is it sufficient for the above equivalence?

No,it's not sufficient. Consider the following example. Let $X = \{a, b, c\}$. Suppose $C(\{a, b, c\}) = \{a, b, c\}$, $C(\{a, b\}) = \{a\}$, $C(\{b, c\}) = \{b\}$, $C(\{a, c\}) = \{c\}$. We can easily check that this example satisfies the Choice Axiom, but it doesn't satisfy the WA. To see this let $A = \{a, b, c\}$, $B = \{a, b\}$. Now notice that $b \in C(A)$, $a \in C(B)$, $a, b \in A \cap B$, but $b \notin C(B)$.