Princeton University, Economics 501
Midterm Examination Solutions, 1998

Prepared by Wojciech Olszewski

Question 1.

a) Consider the descending sequence of sets

$$X = X_1 \supseteq \ldots \supseteq X_k = \emptyset \text{ such that } X_{i+1} = X_i - \text{supp } X_i \, ,$$

where supp X_i stands for the subset of those elements x_i of X_i that $c(X_i)(x_i) > 0$. Then

if i is the greatest index such that $A \subseteq X_i$, then $c(A)(a) = c(X_i)(a)/\Sigma_{b\in A}c(X_i)(b)$.

Proof. I will consider only the case when A intersects X_1, i.e., the support of c(X). Notice first that if c(X)(a) = 0 , then c(A)(a) = 0 . Otherwise, one could consider any $b \in A$ such that c(X)(b) > 0 (The assumption that A intersects X_1 is important here !) and Axiom I would be violated for those a and b. By a similar argument if c(A)(a) = 0 , then c(X)(a) = 0 . That is

$$c(X)(a) = 0 \text{ if and only if } c(A)(a) = 0 \text{ for every } a \in A . \qquad (*)$$

Now observe that

$$c(A)(b)/ c(A)(a) = c(X)(b)/ c(X)(a) \text{ if } c(X)(a) \neq 0 \text{ (or, equivalently if } c(A)(a) \neq 0 \text{).} \quad (**)$$

Indeed,

$$1+ c(A)(b)/c(A)(a) = [c(A)(a) + c(A)(b)]/c(A)(a)$$

= (by Axiom I) =

$$[c(X)(a) + c(X)(b)]/c(X)(a) = 1+ c(X)(b)/c(X)(a) .$$

Finally, (*) and (**) yield $c(A)(a) = c(X)(a)/\Sigma_{b\in A}c(X)(b)$.

b) Consider the probability distribution that assigns 1/3 to every of the following three orderings: (a,b,c) , (c,a,b) , (b,a,c) , and 0 to the other orderings. Then

$$c(X)(a) = c(X)(b) = c(X)(c) = 1/3 ,$$

but C({a,c})(a) = 2/3 , c({a,c})(c) = 1/3 , so Axiom I is violated (according to I one should obtain C({a,c})(a) = c({a,c})(c) = 1/2).

Question 2.

a) This is the indirect utility function of a consumer facing prices p and having initially endowment w. It measures happiness that comes from a given endowment of goods in the world where prices are fixed and equal to p, and the consumer can exchange any amount of goods from her (his) original basket for other goods.

b) $V(\lambda p,w) = \max\{ u(x) : \lambda px = \lambda pw \} = \max\{ u(x) : px = pw \} = V(p,w)$

c) Let $V(p_1,w) \le V^*$ and $V(p_2,w) \le V^*$. Suppose $x^* \in \operatorname{argmax}\{ u(x) : px = pw \}$, where $p = \lambda p_1 + (1-\lambda)p_2$. If $p_1x^* > p_1w$ and $p_2x^* > p_2w$, then $px^* > pw$. So either $p_1x^* \le p_1w$ or, $p_2x^* \le p_2w$, say $p_1x^* \le p_1w$. Then

$$V(p,w) = u(x^*) \le \max\{ u(x) : p_1x = p_1w \} \le V^*.$$

d) **Argument 1.** (preferred by Professor Rubinstein, at leats this sort of arguments). Consider the change of ε (either positive or negative) in p_i . If this change is combined with the change of $\varepsilon(x_i(p,w) - w_i)/p_i$ in w_i, and if you consume again the same vector $x(p,w)$, then you have

$$p_{-i}[w_{-i} - x_{-i}(p,w)] + [p_i + \varepsilon][w_i + \varepsilon(x_i(p,w) - w_i)/p_i - x_i(p,w)] =$$
$$p_{-i}[w_{-i} - x_{-i}(p,w)] + p_i[w_i - x_i(p,w)] + \varepsilon^2(x_i(p,w) - w_i)/p_i = \varepsilon^2(x_i(p,w) - w_i)/p_i$$

money left. It can be either positive or negative. Notice however that ε appears with square, so you are on the budget line up to the first order approximation.

Argument 2. The slope of any indifference curve of V is given by

$$-\partial V(p,w)/\partial p_i/\partial V(p,w)/\partial w_i .$$

Since $V(p,w) = \max\{ u(x) : px = pw \}$, by the envelope theorem

$$\partial V(p,w)/\partial p_i = \lambda(x_i(p,w) - w_i) \text{ and } \partial V(p,w)/\partial w_i = -\lambda p_i.$$

Thus

$$-\partial V(p,w)/\partial p_i/\partial V(p,w)/\partial w_i = (x_i(p,w) - w_i)/p_i .$$

Question 3.

I give the solution of the original question. Under the strengthened version of I, the solution simplifies quite a bit. For example, Proposition 2 is straightforward, and the proof of Proposition 3 reduces significantly.

a) **Consensus**: if all department's members think that someone is (not) an economist, then that individual is found (not) to be an economist. It rules out situations that the aggregated opinion about an individual is independent of department members' opinion.
Independence: the aggregated opinion about an individual is independent of department members' opinion about other individuals. It suggests that "being found an economist by the department" has absolute rather than relative meaning.

b) "The only real economist at Economics Department of Princeton University is …. It really does not matter what those guys from ED of PU think" satisfies I but not C.
"$F(E_1,\ldots,E_n) = E_1 \cap \ldots \cap E_n \cup \{k(E_1,\ldots,E_n)\}$, where $k(E_1,\ldots,E_n)$ is an arbitrary element of $N - (N- E_1) \cap \ldots \cap (N- E_n)$, satisfies C but not I.

c) **Definition 1.** A coalition G is almost decisive for some j if

$$[\ j \in E_i \text{ for all } i \in G \text{ and } j \notin E_i \text{ for all } i \notin G\ ] \text{ implies } [\ j \in F(E_1,\ldots,E_n)\ ]$$

and

$$[\ j \notin E_i \text{ for all } i \in G \text{ and } j \in E_i \text{ for all } i \notin G\ ] \text{ implies } [\ j \notin F(E_1,\ldots,E_n)\ ] .$$

Definition 2. A coalition G is almost decisive if it is almost decisive for every j.

Definition 3. A coalition G is decisive if for every j,

$$[\ j \in E_i \text{ for all } i \in G] \text{ implies } [\ j \in F(E_1,\ldots,E_n)\ ]$$

and

$$[\ j \notin E_i \text{ for all } i \in G] \text{ implies } [\ j \notin F(E_1,\ldots,E_n)\ ] .$$

The proof consists of the following three propositions.

Proposition 1. If G is almost decisive, then G is decisive.

Proof. Suppose that G is not decisive. First consider the case that $j \notin F(E_1,\ldots,E_n)$ although $j \in E_i$ for all $i \in G$. Let $k \neq j$ be an arbitrary element of N. Put

$$E_i^{/} = \{j\} \text{ for } i \in G ,$$

and

$$E_i^{/} = N - \{k\} \text{ if } j \in E_i \text{ and } E_i^{/} = N - \{j,k\} \text{ if } j \notin E_i \text{ for } i \notin G .$$

Since $n > 2$, $E_i^{/}$ is a proper subset of N for every i. By I (Independence), $j \notin F(E_1^{/},\ldots,E_n^{/})$ and by C (Consensus)), $k \notin F(E_1^{/},\ldots,E_n^{/})$. Since $F(E_1^{/},\ldots,E_n^{/}) \neq \emptyset$, it contains some $m \neq j, k$. It violates the assumption that G is almost decisive because $m \notin E_i^{/}$ for all $i \in G$ and $m \in E_i^{/}$ for all $i \notin G$.

Now suppose that $j \notin E_i$ for all $i \in G$ but $j \in F(E_1,\ldots,E_n)$. Let again $k \neq j$ be an arbitrary element of N. Put

$$E_i^{/} = N - \{j\} \text{ for } i \in G ,$$

and

$$E_i^{/} = \{k\} \text{ if } j \notin E_i \text{ and } E_i^{/} = \{j,k\} \text{if } j \in E_i \text{ for } i \notin G .$$

Apply a similar argument.

Proposition 2. If G is almost decisive for some j, then G is almost decisive.

Proof. Take any $k \neq j$. First suppose that $k \in E_i$ for all $i \in G$ and $k \notin E_i$ for all $i \notin G$, and $k \notin F(E_1,\ldots,E_n)$. Consider

$$E_i^{/} = \{k\} \text{ for } i \in G , \text{ and } E_i^{/} = \{j\} \text{for } i \notin G .$$

By I , $k \notin F(E_1^{/},\ldots,E_n^{/})$. Since G is almost decisive for j, also $j \notin F(E_1^{/},\ldots,E_n^{/})$. By C , $m \notin F(E_1^{/},\ldots,E_n^{/})$ for any $m \neq k, j$. Thus $F(E_1^{/},\ldots,E_n^{/}) = \emptyset$, a contradiction.

Now suppose that $k \notin E_i$ for all $i \in G$ and $k \in E_i$ for all $i \notin G$. Consider $E_i^{/} = N - \{k\}$ for $i \in G$, and $E_i^{/} = N - \{j\}$ for $i \notin G$ and show that $F(E_1^{/},\ldots,E_n^{/}) = N$, a contradiction.

Proposition 3. If $\{G_1,G_2\}$ is a partition of G, and G is almost decisive, then either G_1 is almost decisive for some j or G_2 is almost decisive for some j.

Proof. It consists of three steps.

Step 1. Either there is a j such that

$$\begin{aligned} &[\, j \in E_i \text{ for all } i \in G_1 \text{ and } j \notin E_i \text{ for all } i \notin G_1 \,] \text{ implies } [\, j \in F(E_1,\ldots,E_n) \,] \\ &\text{and} \\ &[\, j \in E_i \text{ for all } i \in G_2 \text{ and } j \notin E_i \text{ for all } i \notin G_2 \,] \text{ implies } [\, j \in F(E_1,\ldots,E_n) \,]. \end{aligned} \tag{1}$$

Or there are j and m, $j \neq m$, such that for some $\varepsilon = 1,2$,

$$\begin{aligned} &[\, j \in E_i \text{ for all } i \in G_\varepsilon \text{ and } j \notin E_i \text{ for all } i \notin G_\varepsilon \,] \text{ implies } [\, j \in F(E_1,\ldots,E_n) \,] \\ &\text{and} \\ &[\, m \in E_i \text{ for all } i \in G_\varepsilon \text{ and } m \notin E_i \text{ for all } i \notin G_\varepsilon \,] \text{ implies } [\, m \in F(E_1,\ldots,E_n) \,] . \end{aligned} \tag{2}$$

Indeed, take $j \neq k$ and consider

$$E_i = \{j\} \text{ for } i \in G_1 ,\ E_i = \{k\} \text{ for } i \in G_2 , \text{ and } E_i = N - \{j,k\} \text{ for } i \notin G .$$

Since G is almost decisive only j and k can belong to $F(E_1,\ldots,E_n)$. Suppose first that $F(E_1,\ldots,E_n) = \{j,k\}$ and consider

$$E_i' = \{k\} \text{ for } i \in G_1 ,\ E_i' = \{j\} \text{for for } i \in G_2 , \text{ and } E_i' = N - \{j,k\} \text{ for } i \notin G .$$

By the same argument $F(E_1',\ldots,E_n') \subseteq \{j,k\}$, say $j \in F(E_1',\ldots,E_n')$. Then by I condition (1) is satisfied.

Suppose now that $F(E_1,\ldots,E_n)$ contains only one element, say j. Take $m \neq k,j$ and consider

$$E_i' = \{m\} \text{ for } i \in G_1 ,\ E_i' = \{k\} \text{for for } i \in G_2 , \text{ and } E_i' = N - \{j,m\} \text{ for } i \notin G .$$

Again $F(E_1',\ldots,E_n') \subseteq \{j,m\}$. But if k belonged to $F(E_1',\ldots,E_n')$, then k would belong to $F(E_1,\ldots,E_n)$ by I, and we would have a contradiction. So $F(E_1',\ldots,E_n') = \{m\}$, and (2) is satisfied for $\varepsilon = 1$.

Step 2. If (1) is satisfied, the (2) must be satisfied as well.

Indeed, take k, and m such that $k \neq m$, $k \neq j$ and $m \neq j$, and consider

$$E_i = \{k\} \text{ for } i \in G_1 ,\ E_i = \{m\} \text{ for } i \in G_2 , \text{ and } E_i = N - \{k,m\} \text{ for } i \notin G .$$

By I , $F(E_1,\ldots,E_n) \subseteq \{k,m\}$. Suppose $m \in F(E_1,\ldots,E_n)$. Then (2) is satisfied for $\varepsilon = 2$ by I and (1).

Step 3. If (2) is satisfied, say for $\varepsilon = 1$, then G_1 is almost decisive either for j or for m.

Otherwise consider

$$E_i = N - \{m\} \text{ for } i \in G_1 ,\ E_i = N - \{j\} \text{ for } i \in G_2 , \text{ and } E_i = \{m\} \text{for } i \notin G .$$

If $m \notin F(E_1,\ldots,E_n)$, then by I , G_1 is almost decisive either for m.

Suppose thus that $m \in F(E_1,\ldots,E_n)$. Since G is almost decisive, $F(E_1,\ldots,E_n)$ contains $N - \{j,m\}$. And by (2), $j \in F(E_1,\ldots,E_n)$. So $F(E_1,\ldots,E_n) = N$, a contradiction.