Course: **Econ 501, Princeton University**
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Time: **3 hours (no extensions)**
Instructions: **Answer the following two questions. Be concise and accurate.**

Problem 1

Consider a consumer with a preference relation in a world of two goods: X (an aggregated consumption good) and M ("membership in a club", for example), which can be consumed or not. In other words, the consumption of X can be any non-negative real number while the consumption of M must be either 0 or 1.

Assume that consumer preferences are strictly monotonic, continuous and satisfy property E: For every x there is y such that $(y,0) \succ (x,1)$ (that is, there is always some amount of money which can compensate for the loss of membership).

■A) Show that any consumer's preference relation can be represented by a utility function of the type

$$u(x,m) = \begin{cases} x & if \ \ m=0 \\ x+g(x) & if \ \ m=1 \end{cases}$$

(Answer)

Construct $u(x,m)$ as follows:

(1) Let $u(x,0) = x$ for all $x \geq 0$.

(2) Take any x, Find a value $h(x)$ such that $(x,1) \sim (h(x),0)$. Then let $g(x) = h(x) - x$ and $u(x,1) = x + g(x) (= h(x))$

Notice that such $h(x)$ always exists and it is unique. This is because $(0,0) \prec (x,1)$ by monotonicity and $(y,0) \succ (x,1)$ for some y by E so continuity implies that $(x,1) \sim (y',0)$ for some y'. Also, it must be unique because of monotonicity.

Next, let's verify this u actually represents $\succcurlyeq$,

Case 1 $(x,0) \succcurlyeq (x',0)$, it is equivalent to $u(x,0) = x \geq x' = u(x',0)$

Case 2 $(x,1) \succcurlyeq (\preccurlyeq)(x',0)$, it is equivalent to $(h(x),0) \succcurlyeq (\preccurlyeq)(x',0) \Leftrightarrow u(x,1) \geq (\leq) u(x',0)$

Case 3 $(x,1) \succcurlyeq (x',1)$ it is equivalent to $(h(x),0) \succcurlyeq (h(x'),0) \Leftrightarrow u(x,1) \geq u(x',1)$

Therefore, u defined above actually represents $\succcurlyeq$.

■B) (Less easy) Show that the consumer's preference relation can also be represented by a utility function of the type $u(x,m) = \begin{cases} f(x) & if \ \ m=0 \\ f(x)+v & if \ \ m=1 \end{cases}$

(Answer)

Let $h(x)$ be the value such that $(x,1) \sim (h(x),0)$. By continuity, monotonicity, and E, such h is a well-defined and strictly increasing in x. Define $h^n(x) = h^{n-1}(h(x))$. and $h^0(x) = x$. Since $(h^n(x),1) \sim (h^{n+1}(x),0)$, monotonicity implies $h^{n+1}(x) > h^n(x)$ for all n.

Construct $u(x,m)$ as follows:

(1) Let $f(0) = 0$ (Just a normalization)

(2) Let $f(x)$ for $x \in (0,h(0)]$ some arbitrary increasing function and let $v = f(h(0))$.

(3) Second, define $f(x)$ for $x \in (h(0),h^2(0)]$ as follows. Since h is an increasing function, h^{-1} exists so we can define $f(x) = v + f(h^{-1}(x))$. Since h^{-1} is increasing, we have $h^{-1}(x) \in (0,h(0)]$, and $f(h^{-1}(x))$ has been already defined in the previous step. Hence, this definition works well. Since f is increasing in $x \in [0,h(0)]$, and h^{-1} is increasing, f is increasing also in $x \in [0,h^2(0)]$, and $f(h^2(0)) \ = v + f(h^{-1}(h^2(0)) \ = v + f(h(0)) \ = 2v$.

(4) Continue this step forever. That is: if $f(x)$ has been already defined for $x \in [0, h^{n-1}(0)]$ such that $f(x)$ is increasing in this region, $f(x) = v + f(h^{-1}(x))$ for $x \in [h(0), h^{n-1}(0)]$ and $f(h^k(0)) = kv$ for all $k \leq n-1$, then define $f(x)$ for $x \in (h^{n-1}(0), h^n(0)]$, such that $f(x) = v + f(h^{-1}(x))$. It can be verified in the same way as in the previous step that this is well defined, f is increasing in $x \in [0, h^n(0)]$ and $f(h^n(0)) = nv$.

(5) Define $u(x,0) = f(x)$ and $u(x,1) = f(x) + v$.

We have to make sure that this f is actually defined for all $x \geq 0$. Since $\{h^n(0)\}$ is an increasing sequence, $f(x)$ is not defined twice or more in the above steps. Therefore, it is sufficient to show that $\lim_{n\to\infty} h^n(0) = \infty$.

Suppose this is not, since $\{h^n(0)\}$ is an increasing sequence, it must be $\lim_{n\to\infty} h^n(0) = K$ and $h^n(0) < K$ for all n. This means that $(h^n(0),1) \sim (h^{n+1}(0),0) \prec (K,0)$ for all n where the first indifference comes from the definition of h and the second preference comes from monotonicity. Take a limit of the both sides, we have $(K,1) \preccurlyeq (K,0)$ because of continuity but this contradicts monotonicity.

Finally, we need to confirm that this f actually represents $\succcurlyeq$. Notice that u represents $\succcurlyeq$ correctly between $(x,0)$ and $(x',0)$ (or $(x,1)$ and $(x',1)$) because f is an increasing function. Therefore, we need to show that $(x,1) \succcurlyeq (x',0)$ if and only if $f(x) + v \geq f(x')$

Suppose $(x,1) \succcurlyeq (x',0)$. Then, by the definition of h, $h(x) \geq x'$. Since f is increasing, we have $u(h(x),0) = f(h(x)) \geq f(x') = u(x',0)$. By construction of f, we have $f(h(x)) = v + f(h^{-1}(h(x))) = v + f(x) = u(x,1)$. Therefore, $u(x,1) \geq u(x',0)$. The same argument can be applied for the case when $(x,1) \preccurlyeq (x',0)$. Hence, we conclude that u defined above actually represents $\succcurlyeq$.

■C) Explain why continuity and strong monotonicity (without E) are not sufficient for A.

(Answer)

Let's consider the lexicographic preference $\gtrsim$ which gives a priority to M. (so $(x,m) \succ (x',m')$ if and only if $m > m'$ or "$m = m'$ and $x > x'$").

This is clearly strictly monotonic both in x and m. Notice that this is a continuous preference. To see this, take any $(x,m) \succ (x',m')$. Suppose $m > m'$ (so $m = 1$ and $m' = 0$. This is crucial to have a continuity and you should understand why this proof does not work if $X = R^2$). If we take a enough small $\varepsilon > 0$, then $(x'',m'') \in B_\varepsilon((x,m))$ implies $m = 1$ and $(x''',m''') \in B_\varepsilon((x',m'))$ implies $m = 0$. Therefore, $(x'',m'') \succ (x''',m''')$. When $m = m'$ (so $x > x'$) it is true that if we take a enough small $\varepsilon > 0$, then $(x'',m'') \in B_\varepsilon((x,m))$ and $(x''',m''') \in B_\varepsilon((x',m'))$ imply $m = m' = m'' = m'''$ but $x'' > x'''$ so we have $(x'',m'') \succ (x''',m''')$. Therefore, we conclude that $\gtrsim$ is continuous.

However, $\gtrsim$ cannot be represented by a utility function with the form given in A) if $\gtrsim$ does not satisfy E. To see this, suppose $\gtrsim$ can be represented as in A). Then for some x, $u(x,1) > u(y,0)$ for all $y \geq 0$. This means that $x + g(x) > y$ for all $y \geq 0$. Clearly, this is impossible.

■D) Compute the consumer's demand function.

(Answer)

By monotonicity, the consumer always spends all wealth on X or M so his choice is simply between "Buying M and spending all the remaining wealth on X" and "Spending all the wealth on X" if $p_m \leq w$. If $p_m > w$, he has no choice except "spending all the wealth on X". Therefore, his demand function is characterized by

$$(x(p,w), m(p,w)) = \begin{cases} (w/p_x, 0) & \text{if } (w/p_x,0) \succcurlyeq ((w-p_m)/p_x, 1) \text{ or } p_m > w \\ ((w-p_m)/p_x, 1) & \text{if } (w/p_x,0) \preccurlyeq ((w-p_m)/p_x,1) \text{ and } p_m \leq w \end{cases}$$

where p_i is the price of good i, and w is the wealth.

■E) Taking the utility function to be of the form described in part (A), compute the consumer's indirect utility function. For the case that the function g is differnetiable verify the Roy equality in respect to commodity M.

(Answer)

If the consumer's utility function is given by a differentiable utility function as in part (A), then his/her indirect utility function is

$$v(p,w) = \begin{cases} w/p_x & \text{if } w/p_x \geq (w-p_m)/p_x + g((w-p_m)/p_x) \text{ or } p_m > w \\ (w-p_m)/p_x + g((w-p_m)/p_x) & \text{if } w/p_x \leq (w-p_m)/p_x + g((w-p_m)/p_x) \text{ and } p_m \leq w \end{cases}$$

In the first case,

$$\frac{\partial v/\partial p_m}{\partial v/\partial w} = 0 = m(p,w)$$

and in the second case,

$$\frac{\partial v/\partial p_m}{\partial v/\partial w} = -\frac{(-1)\cdot(1/p_x) + g'\cdot(-1/p_x)}{1/p_x + g'\cdot(1/p_x)} = 1 = m(p,w)$$

so the Roy equality holds.

Problem 2

The standard economic choice model assumes that choice is made from a *set*. Let us construct a model where the choice is assumed to be from a *list*.

Let X be a finite "grand set". A *list* is a non-empty finite vector of elements in X. In this problem, consider a *choice function* C to be a function which assigns to each vector $L =< a_1,\ldots,a_K >$ a single element from $\{a_1,\ldots,a_K\}$. (Thus, for example, the list $< a,b >$ is distinct from $< a,a,b >$ and $< b,a >$). For all $L_1,\ldots,L_m$ define $< L_1,\ldots,L_m >$ to be the list which is the concatenation of the m lists. (Note that if the length of L_i is k_i the length of the concatenation is $\Sigma_{i=1,..,m}k_i$). We say that L' *extends* the list L if there is a list M such that $L' =< L,M >$.

We say that a choice function C satisfies property I if for all $L_1,\ldots,L_m$
$C(< L_1,\ldots,L_m >) = C(< C(L_1),\ldots,C(L_m) >)$.

(Notation) Let me define some notations. Let $L =< a_1,\ldots,a_K >$. We define $S(L)$ as a set which is consisted of all the elements which is in L. Formally, it is defined as $S(L) = \{a \in X \mid a = a_k$ for some $k \in \{1,\ldots,K\}\}$.

■A) Interpret property I. Give two (distinct) examples of choice functions which satisfy I and two examples of choice functions which do not.

(Answer)

I requires that the decision maker makes the same decision

(i) when he was given a list which is a concatination of some sublists

and

(ii) when he was given a shorten list such that each sublist in the original list is replaced with one element which he would choose when he was given the sublist as a whole list but the order of the elements must be the same as the order of the corresponding sublists in the original.

Example 1 (satisfying I): Consider a rational choice function. That is: the decision maker has a strict preference $\succ$ over X and he chooses an element from the list which is the $\succ$-best. (i.e. $C(L) = a$

such that $a \in L$ and $a \succ a'$ for any $a' \in S(L)\backslash\{a\}$) Suppose $a = C(< L_1,\ldots,L_m >)$, then (*) $a \succ a'$ for all $a' \in (S(L_1) \cup \cdots \cup S(L_m))\backslash\{a\}$. Since $a \in S(L_k)$ for some k and $S(L_k) \subset S(L_1) \cup \cdots \cup S(L_m)$, (*) implies $a \succ a'$ for all $a' \in S(L_k)\backslash\{a\}$ so $a = C(L_k)$. Note also that $S(< C(L_1) \cup \cdots \cup C(L_m) >) \subset S(L_1) \cup \cdots \cup S(L_m)$. so again (*) implies $a = C(< C(L_1),\ldots,C(L_m) >)$. Therefore, C satisfies I.

Example 2 (satisfying I): The decision maker chooses the first element in the list. (i.e. $C(< a_1,\ldots,a_K >) = a_1$)) Then, $C(< L_1,\ldots,L_m >)$ =”the first component in $< L_1,\ldots,L_m >$”=”the first component of L_1”$= C(L_1) = C(< C(L_1),\ldots,C(L_m) >)$ so C satisfies I.

Example 3 (satisfying I) : The decision maker has a strict preference $\succ$ over X and a satisfactory element x. If the list contains an element which is weakly preferred to x, then, among them, he chooses the one which appears the first. If there is no such an element, chooses the $\succ$ −best element in the list. (i.e. if there exists $a \in S(L)$ such that $a \succcurlyeq x$, then $C(< a_1,\ldots,a_K >) = a_i$ where $i = \min\{k \in \{1,\ldots,K\} \mid a_k \succcurlyeq x\}$ and otherwise the same as in Example 1)

This satisfies I. To see why, if there is no $a \in S(L)$ such that $a \succcurlyeq x$, then apply the same proof as in Example 1. If there is such an element, let a be the one which appears the first among them in L. Suppose $L =< L_1,\ldots,L_m >$, then we can always find n such that $a \in S(L_n)$, a is the first element in the L_n which is weakly preferred to x, and $a' \prec x$ for any $a' \in S(L_1),\ldots,S(L_{n-1})$. Therefore, $C(L_{n'}) \prec x$ for all $n' = 1,\ldots,n-1$ and $C(L_n) = a$. Hence, $C(L) = a = C(< C(L_1),\ldots,C(L_m) >)$.

Example 4 (violating I) The decision maker has a strict preference $\succ$ over X and he chooses an element from the list which is the $\succ$-second-best. (i.e. if $|S(L)| \geq 2$, then $C(L) = a$ such that $a \in S(L)$, $\exists! b \in S(L)$ such that $b \succ a$. If $|S(L)| = 1$, $C(L) = S(L)$.) This violates I because, for instance, let $x \succ y \succ z$, then
$C(< x,y,z >) = y \neq z = C(< y,z >) = C(< C(< x,y >),C(< z >) >)$.

Example 5 (violating I): The decision maker chooses the second element in the list (if more than one element are in the list.). Then,

$C(< a,b,c,d >) = b$ but $C(< C(< a,b >),C(< c,d >) >) = C(< b,d >) = d$ so C violates I.

Example 6 (violating I): The decision maker chooses the elements which appears the list most often. In case of a tie, chooses one from them which appears the first. Then,
$C(< a,a,b,b,b,b,a,a,b >) = b$ but
$C(< C(< a,a,b >),C(< b,b,b >),C(< a,a,b >) >) = C(< a,b,a >) = a$, this is a violation of I.

■B) Define formally the following two properties of a choice function:

Order Invariance: A change in the order of the elements of the list does not alter the choice and

Duplication Invariance: Deleting an element which appears in the list elsewhere does not change the choice.

Characterize the choice functions which satisfy Order Invariance, Duplication Invariance and condition I.(Actually, Duplication Invariance is redundant...)

(Answer)

Order Invariance(OI): Given $L =< a_1,\ldots a_K >$. For any permutation P (i.e. P is a one-to-one function from $\{1,\ldots K\}$ to itself), $C(L) = C(L')$ where $L' =< a_{P(1)},\ldots a_{P(K)} >$.

Duplication Invariance(DI): Given $L =< a_1,\ldots a_K >$. Suppose there exist $i \neq j$ such that $a_i = a_j$, Define a new list $L' =< a'_1,\ldots,a'_{K-1} >$ such that $a'_k = a_k$ if $k < i$ and $a'_k = a_{k+1}$. for $k \geq i$ (so L' can be obtained from L by deleting a_i but keeping its order elsewhere). Then $C(L) = C(L')$

Claim: If a choice function C satisfies (OI) and I, then there exists a strict preference $\succ$ over X such that for any list L, $C(L) = a$ where $a \in S(L)$ and $a \succ a'$ for all $a' \in S(L)\backslash\{a\}$. Conversely, if there exists such a strict preference, then C satisfies these three properties.

Proof:

(The first part)

Suppose C satisfies (OI), and I. Define $\succ$ as follows.

(1) Not $a \succ a$ for all $a \in X$

(2) For $x \neq y, x \succ y$ if and only if $C(< x,y >) = x$

We argue that $\succ$ is actually a strict preference over X. To see this:

(Asymetricity) By (1) not $x \succ x$. If $x \succ y$, then by (2), $x = C(< x,y >) = C(< y,x >)$ where the second equality comes from (OI). Therefore not $y \succ x$.

(No two distinct elements are indifferent) If $x \neq y$, then $x = C(< x,y >)$ or otherwise, $y = C(< x,y >) = C(< y,x >)$ by (OI) so we have $x \succ y$ or $y \succ x$ (but not both).

(Negative Transitivity) Suppose $x \succ y$ (so $C(< x,y >) = x$ by (2))and take any z. If $z = x$ or y, negative transitivity holds trivially. Assume $z \neq x,y$, but not $x \succ z$, and not $z \succ y$. Since not $x \succ z$, $C(< z,x >) = C(< x,z >) = z$ where the first equality is by (OI) so we have $z \succ x$. Similarly, we have $y \succ z$. By (2), we have $C(< z,x >) = z$ and $C(< y,z >) = y$. Applying I and (OI), we have

$$C(< x,y,z >) = C(< C(< x,y >), C(< z >) >) = C(< x,z >) = C(< z,x >) = z \text{ and}$$
$$C(< x,y,z >) = C(< C(< x >), C(< y,z >) >) = C(< x,y >) = x$$

This is a contradiction so $x \succ z$ or $z \succ y$.

Next, we will show that for all L, $C(L) = a$ where $a \in S(L)$ and $a \succ a'$ for all $a' \in S(L) \backslash \{a\}$. Such a always exists and is unique because $S(L)$ is a finite set and no two distinct elements are indifferent. Let $L^0 = L$ and given $L^{n-1} =< a_1, \ldots, a_K >$ define L^n as follows.

If K is even, $L^n =< C(< a_1,a_2 >), \ldots C(< a_{K-1},a_K >) >$ and if K is odd $L^n =< C(< a_1,a_2 >, \ldots, C(< a_{K-2},a_{K-1} >), a_K >$

By I, $C(L^n) = C(L^{n+1})$ for all n so $C(L) = C(L^n)$ for any n.

By I and the construction of $\succ$, $a = C(< a,b >) = C(< b,a >)$ for all $b \in S(L^n) \subset S(L)$. Therefore, $a \in S(L^n)$ for all n. Since for large N, L^N contains only one elements, we have $L^N =< a >$ so $C(L^N) = a$. Hence, we have $C(L) = C(L^N) = a$.

(The second part)

We have already shown that such C satisfies I in part a). It is obvious that C does not depend on the order of elements so satisfies (OI).

(Remark: Actually, (DI) is redundant because it is implied by I and (OI). To see this, If there is an element which appears twice, then move one of them next to the other and apply a choice function to the two in advance, which makes the two same element into a single element. By I and (OI), the outcome remains the same.)

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Assume now that in the back of the decision maker's mind is a value function u defined on the set X (such that $u(x) \neq u(y)$ for all $x \neq y$). For any choice function C define $v_C(L) = u(C(L))$.

We say that C *accommodates a longer list* if whenever L' extends L, $v_C(L') \geq v_C(L)$ and there is a list L' which extends a list L for which $v_C(L') > v_C(L)$.

■C) Give two interesting examples of a choice function which accommodates a longer list.

(Answer) Assume L' extends L.

Example 1: C is the rational choice function with a preference represented by u. (so $C(L) = \arg\max_{x \in S(L)} u(x)$). Since $S(L) \subset S(L')$, $v_C(L) = \arg\max \max_{x \in S(L)} u(x) \leq \max_{x \in S(L')} u(x) = v_C(L')$. Let $u(y) > u(x)$. Then $v_C(< x >) = u(x) < u(y) = v_C(< x,y >)$.

Example 2: C chooses an element from the first three elements which maximizes u. Then S("the first 3 elements of L") $\subset$ ("the first 3 elements of L'") so $v_C(L) \leq v_C(L')$. Suppose

$u(x) < u(y) < u(z)$, then $v_C(< x,y >) = u(y) < u(z) = v_C(< x,y,z >)$

(Remark, this does not satisfies I)

Example 3: (the same example as the Example 3 in part A)) The decision maker has a satisfactly level $\underline{u}$ and among those yielding a higher or equal value u, he chooses the one which appears the first. If there is no such an element, chooses the element which maximizes u. (i.e. if $u(a) \geq \underline{u}$ for some $a \in S(L)$, $C(< a_1,\ldots,a_K >) = a_i$ where $i = \min\{k \in \{1,\ldots,K\} \,|\, u(a_k) \geq \underline{u}\}$ and if $u(a_k) < \underline{u}$ for all k, $C(< a_1,\ldots,a_K >) = \arg\max_{x \in S(L)} u(x)$)

This accommodates a longer list because:

If $L =< a_1,\ldots,a_K >$ contains elements which give a higher value than $\underline{u}$, let a be the one which appears the first among them, then a also appears the first in L' among them so $v_C(L) = u(a_i) = v_C(L')$

If L does not contain such an element, but L' contains such an element, then $v_C(L') \geq \underline{u} > v_C(L)$ and when L' does not contain such an element neither, then we can show $v_C(L') \geq v_C(L)$ in the same way as in Example 1.

Suppose $u(x) < u < u(y)$, then $v_C(< x >) = u(x) < u(y) = v_C(< x,y >)$.

(Remark, this does not satisfy (OI))

Example 4: C chooses an element a_i which maximizes $(1/2)^k u(a_k)$ from $< a_1,\ldots,a_K >$ (assuming that u always takes a positive value.)

This accommodates a longer list. When $C(< a_1,\ldots,a_K >) = C(< a_1,\ldots,a_K,a_{K+1},\ldots,a_{K'} >)$, then his utility remains the same. When $C(< a_1,\ldots,a_K >) = a_i \neq a_j = C(< a_1,\ldots,a_K,a_{K+1},\ldots,a_{K'} >)$, then $j \geq K+1$ so $i < j$. Since $(1/2)^j u(a_j) > (1/2)^i u(a_i)$, we have $u(a_j) > u(a_i)$ so his utility increases by extending the list. Hence, $v_C(L) \leq v_C(L')$. Suppose $u(x) = 15$, $u(y) = 3$, and $u(z) = 5$, then $v_C(< y,z >) = 3 < 15 = v_C(< y,z,x >)$

■D) Give two interesting examples of choice functions which satisfy property I but which do not accommodate a longer list.

(Answer) Assume L' extends L.

Example 1: C always chooses the first element in the list.

It satisfies I as is shown in the part A). However, it does not accommodate a longer list because the first elements of L is the same as that of L' so for any L and it extension L', $v_C(L) = v_C(L')$

Example 2: When L contains at least one element which yields a utility higher than or equal to $\underline{u}$, C chooses the element which maximizes u. If not, C picks an element which appears the last in L.

This satisfies I. (If L contains an element whose utility is higher than $\underline{u}$, apply the same proof as for a rational choice function. If not, apply the same proof as for Example 1.) However, it does not accommodate a longer list because if $u(y) < u(x) < \underline{u}$, then $v_C(< x >) = u(x) > u(y) = v_C(< x,y >)$

Example 3: C chooses the element from the list which minimizes u.

C satisfies I (can be shown in the same way as for a rational choice function) but it does not accommodate a longer list because if $u(x) > u(y)$, $v_C(< x >) = u(x) > u(y) = v_C(< x,y >)$.

Example 4: C chooses the element which maximizes a value function u' (also defined over X) which is distinct from u. (Example 3 is a special case of this example.)

C satisfies I (can be shown in the same way as for a rational choice function) but this does not accommodate a longer list because if $u(x) > u(y)$ and $u'(x) < u'(y)$, then $v_C(< x,y >) = u(y) < u(x) = v_C(< x,y >)$.

(Remark: One of the interpretaion of Example 4 is as follows. You always ask your mother to make a desicion and she chooses an element from the list so as to maximizes her (well defined) utility function, which is different from yours.)

(Remark: The choice functions in Example 3 and 4 can be interpreted as a rational choice function so some of you may be confused why Example 1 in part C) accommodates a longer list but the two in part D) does not. This is because we have already fixed a particular preference (or utility function) when we define the concept of ”accommodating a longer list” so among those who can be interpreted as a rational choice function, only the one which maximizes that particular preference does accommodate a longer list.)